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Note on Quasi-injective Modules

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Let A be a ring with identity 1 and ${}_AM$ be a (unital) semisimple left A-module. Then the following density theorem is well known.

Let $D = End({}_AM)$ (the endomorphism ring of ${}_AM$ operating on the opposite side of the scalar). Then for any $\phi \in End(M_D)$ and $m_1, m_2, \cdots, m_n \in M$, there exists $a \in A$ such that $\phi(m_i) = am_i$ $(i = 1, 2, \cdots, n)$.

In connection with this theorem, we prove the following Proposition.

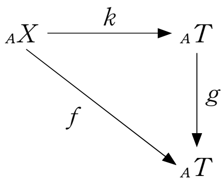
Proposition. Let ${}_AT$ be a quasi-injective left A-module, $M = Soc({}_AT)$ (the socle of ${}_AT$, and assume that $M \neq 0$) and $D = End({}_AT)$. Then

(1) M_D is semisimple as a right D-module;

(2) if A is commutative, then for any $m_1, m_2, \cdots, m_n \in M$ and $\phi \in End(T_D)$, there exists $a \in A$ such that $\phi(m_i) = am_i$ $(i=1, 2, \cdots, n)$;

(3) if ${}_AT$ is an essential extension of ${}_AM$, then for any $m_1, m_2, \cdots, m_n \in M$ and $\phi \in End(T_D)$ (assume that $\phi(m_i) \neq 0$ for some i), there exists u and a in A such that $u\,\phi(m_i) = am_i$ $(i=1, 2, \cdots, n)$ where $u\,\phi(m_i) \neq 0$ for some i.

${}_AT$ is said to be quasi-injective in case for each monomorphism k: ${}_AX \to {}_AT$ and for each homomorphism f: ${}_AX \to {}_AT$, there exists g: ${}_AT \to {}_AT$ such that the following diagram is commutative ([1], p. 191).



Proof of Proposition

(1) Let $D' = End({}_AM)$. Then $M_{D'}$ is semisimple ([3], p. 125) and since ${}_AT$ is quasi-injective, the map $D \to D'$ $(d \to d|M)$ is surjective. So, any D-submodule of M is also D'-submodule and hence M_D is semisimple.

(2) In case A is commutative, D is an A-algebra. Hence for any $\phi \in End(T_D)$, ϕ is also an A-homomorphism. So, $\phi(M) \subset M$. Moreover, since for any $m \in M$ and $d' \in D'$, there exists $d \in D$ such that $\phi(md') = \phi(md) = \phi(m)\,d = \phi(m)\,d'$. Hence ϕ is a D'-homomorphism as well. Therefore by the density theorem, there exists $a \in A$ such that $\phi(m_i) = am_i$ $(i = 1, 2, \cdots, n)$.

(3) Let $m_1, m_2, \cdots, m_n \in M$ and $\phi \in End(T_D)$, and assume that $\phi(m_1) \neq 0$. By the assumption, there exists $u_1 \in A$ such that $0 \neq u_1\,\phi(m_1) \in M$. If $u_1\,\phi(m_2) \in M$, then putting $u_2 = 1$, we have $0 \neq u_2\,u_1\,\phi(m_1) \in M$ and $u_2\,u_1\,\phi(m_2) \in M$. If $u_1\,\phi(m_2) \notin M$, then there exists $u_2 \in A$ such that $0 \neq u_2\,u_1\,\phi(m_2) \in M$ and $u_2\,u_1\,\phi(m_1) \in M$. By continuing this process, we can obtain u in A such that $u\,\phi(m_i) \in M$ $(i = 1, 2, \cdots, n)$. Since M_D is semisimple, there exists $h \in End(M_D)$ such that the following diagram is commutative

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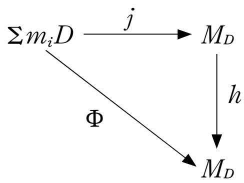

where $\Phi(x) = u\ \phi(x)$ ($x \in \Sigma m_i D$) and j is the inclusion map. As h is a D'-homomorphism, by the density theorem, we have $h(m_i) = am_i$ and $u\ \phi(m_i) = am_i$ ($i = 1, 2, \cdots, n$).

Some examples

Here, in connection with the above Proposition, we give some examples which are QF-modules in the sense of [2]. A bimodule ${}_AQ_B$ is said to be QF if ${}_AQ$ and Q_B are faithful, and for any simple modules ${}_AX$ and Y_B, $Hom({}_AX, {}_AQ)_B$ and ${}_AHom(Y_B, Q_B)$ are simple or zero.

(1) Let Z be the ring of integers and p be a prime number. We put ${}_ZT = E({}_ZZ/(p))$ (an injective hull of ${}_ZZ/(p)$) and $D = End({}_ZT)$. Then it holds that $Soc({}_ZT) = Z/(p) = Soc(T_D)$, and for any $\overline{z} \in Z/(p)$ and $\phi \in End(T_D) \subset End(T_Z)$, $\phi(\overline{z}) = z_0\overline{z}$ where $\phi(\overline{1}) = \overline{z}_0 \in Z/(p)$. In this case, as ${}_ZT$ is divisible, ${}_ZT$ is faithful. Moreover for a prime number p', if $p' = p$, then $Hom({}_ZZ/(p'), {}_ZT)_D = Z/(p)_D$ is simple and if $p' \neq p$, then $Hom({}_ZZ/(p'), {}_ZT)_D = 0$. For any maximal right ideal D_0 of D, if ${}_ZHom(D/D_{0D}, T_D)$ is not zero, ${}_ZHom(D/D_{0D}, T_D)$ is isomorphic to ${}_ZZ/(p)$. Hence ${}_ZHom(D/D_{0D}, T_D)$ is simple and ${}_ZT_D$ is a QF-module.

(2) Let Z be the ring of integers, p_1 and p_2 be distinct prime numbers and Q be the fields of rational numbers. We put ${}_ZT = E({}_ZZ/(p_1)) \oplus E({}_ZZ/(p_2)) \oplus Q$ (external direct sum as Z-modules) and $D = End({}_ZT)$. Although ${}_ZT$ is not an essential extension of $Soc({}_ZT)$, ${}_ZT$ is an injective hull of $Z/(p_1) \oplus Z/(p_2) \oplus Z$, and $Soc({}_ZT) = Z/(p_1) \oplus Z/(p_2)$. Let $\overline{z_1} \in Z/(p_1)$, $\overline{z_2} \in Z/(p_2)$ and $\phi \in End(T_D)$, and assume that $\phi(\overline{z_1}) \neq 0$ and $\phi(\overline{z_2}) \neq 0$. As $p_1 \neq p_2$, we find that $\phi(Z/(p_1)) = Z/(p_1)$ and $\phi(Z/(p_2)) = Z/(p_2)$. By taking $\overline{u} \in Z/(p_1)$ and $\overline{v} \in Z/(p_2)$ such as $\overline{u}\ \overline{p_2}\ \overline{z_1} = \phi(\overline{z_1})$ and $\overline{v}\ \overline{p_1}\ \overline{z_2} = \phi(\overline{z_2})$, we have $\phi(\overline{z_1}) = (u\ p_2 + v\ p_1)\ \overline{z_1}$ and $\phi(\overline{z_2}) = (u\ p_2 + v\ p_1)\ \overline{z_2}$.

Finally, we will see that ${}_ZT_D$ is a QF-module. An element of D is of the form

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} & 0 \\ \lambda_{21} & \lambda_{22} & 0 \\ f_1 & f_2 & q \end{pmatrix}$$

where $\lambda_{ij} \in Hom({}_ZE(Z/(p_i)), {}_ZE(Z/(p_j)))$, $f_i \in Hom({}_ZQ, {}_ZE(Z/(p_i)))$ and $q \in Q$ ($i=1,2$; $j=1,2$). For any simple submodule $Y_D \subset T_D$, let $0 \neq (x_1, x_2, w) \in Y_D$. If $w \neq 0$, for any $(\xi_1, \xi_2, \eta) \in T$, there exist $f_i \in Hom({}_ZQ, {}_ZE(Z/(p_i)))$ such that $(w)f_i = \xi_i$ ($i=1,2$). Hence we obtain the formula

$$(x_1, x_2, w) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ f_1 & f_2 & \eta/w \end{pmatrix} = (\xi_1, \xi_2, \eta),$$

and $(x_1, x_2, w)\ D = T$. But as T_D is not simple, it must be $w = 0$. It is easily seen that if $x_i \neq 0$, then $Y_D = Z/(p_i)$ ($i=1,2$). (The case both $x_1 \neq 0$ and $x_2 \neq 0$ does not occur.) Therefore we have $Soc(T_D) = Z/(p_1) \oplus Z/(p_2) = Soc({}_ZT)$, and in the same way as (1), for any prime number p' and for any maximal right ideal $D_0 \subset D$, $Hom({}_ZZ/(p'), {}_ZT)_D$ and ${}_ZHom(D/D_{0D}, T_D)$ are simple or zero.

References

[1] Anderson F.W. and Fuller K.R. , 1974. Rings and Categories of Modules, Springer-Verlag, NewYork.

[2] Azumaya G., 1959. A duality theory for injective modules (Theory of Quasi-Frobenius Modules), Amer. J. Math., 81: 249-278.

[3] Jacobson N., 1956. Structure of Rings, Amer. Math. Soc. Coll. Pub. vol. 37.

要　約

完全可約加群の density theorem に関連して，quasi-injective 加群の socle について調べ，QF 加群に関連した例について述べた。