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A Note on Separable Modules

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Throughout this note, let A/B be a ring extension. A/B is called a separable extension if the map $\pi' : {}_AA \otimes_B A_A \longrightarrow {}_AA_A$ $(a_1 \otimes a_2 \longrightarrow a_1a_2)$ splits. That is, there exists an A-A-homomorphism $k' : {}_AA_A \longrightarrow {}_AA \otimes_B A_A$ such that $\pi' k' = id_A$ (the identity map of A). If we denote $k'(1) = \sum_i u_i \otimes v_i$, $\sum_i u_i v_i = 1$ and $\sum_i a u_i \otimes v_i = \sum_i u_i \otimes v_i a$ for any $a \in A$.

We assume all rings have the identities 1, all subrings contain the identity of the over ring, and modules are unitary.

Let ${}_AP_A$ be an A-A-bimodule. As a generalization of a separable extension, we call ${}_AP_A$ a separable module over B if it satisfies the following properties:

(1) There exists a surjective A-A-homomorphism $\phi : {}_AP_A \longrightarrow {}_AA_A$ which splits. That is, there exists an A-A-homomorphism $\varepsilon : {}_AA_A \longrightarrow {}_AP_A$ such that $\phi\varepsilon = id_A$.

(2) There exists a surjective A-A-homomorphism $\pi : {}_AP \otimes_B P_A \longrightarrow {}_AP_A$ which splits. That is, there exists an A-A-homomorphism $k : {}_\mathrm{A}\mathrm{P}_\mathrm{A} \longrightarrow {}_\mathrm{A}\mathrm{P} \otimes_\mathrm{B} \mathrm{P}_\mathrm{A}$ such that $\pi k = id_P$.

(3) The following diagram is commutative:

$$\begin{array}{ccc} {}_AP \otimes_B P_A & \xrightarrow{\ \pi\ } & {}_AP_A \\ \phi\otimes\phi \downarrow & & \downarrow \phi \\ {}_AA \otimes_B A_A & \xrightarrow[\ \pi'\ ]{} & {}_AA_A. \end{array}$$

That is, $\phi\pi = \pi'(\phi \otimes \phi)$ where $(\phi \otimes \phi)(p_1 \otimes p_2) = \phi(p_1) \otimes \phi(p_2)$.

Clearly, if A/B is a separable extension, then ${}_AA_A$ is a separable module, and if a separable module ${}_AP_A$ exists, because $\pi'(\phi \otimes \phi)k\varepsilon = \phi\pi k\varepsilon = id_A$, A/B is a separable extension. Here, we will note some examples of separable modules.

Example 1. Let A/B be a separable extension and ${}_AP_A$ be a separable module. Then ${}_AP \otimes_B P_A$ is a separable module.

Proof. Let ϕ, ε, π, and k be maps that satisfy the properties (1), (2), and (3). we define maps

$\boldsymbol{\Phi} : {}_AP \otimes_B P_A \longrightarrow {}_AA_A$ $(p \otimes q \longrightarrow \phi(p)\phi(q))$,

$\boldsymbol{E} : {}_AA_A \longrightarrow {}_AP \otimes_B P$ $(a \longrightarrow (\varepsilon \otimes \varepsilon)k'(a) = (\varepsilon \otimes \varepsilon)(\sum_i a u_i \otimes v_i))$,

$\boldsymbol{\Pi} : (P \otimes_B P) \otimes_B (P \otimes_B P) \longrightarrow P \otimes_B P$ $(p \otimes q \otimes p' \otimes q' \longrightarrow \pi(p \otimes q) \otimes \pi(p' \otimes q'))$,

and $\boldsymbol{K} : P \otimes_B P \longrightarrow (P \otimes_B P) \otimes_B (P \otimes_B P)$ $(p \otimes q \longrightarrow k(p) \otimes k(q))$.

Then $\boldsymbol{\Phi}$, $\boldsymbol{E}$, $\boldsymbol{\Pi}$, and $\boldsymbol{K}$ satisfy properties (1), (2), and (3).

Similarly, ${}_AP \otimes_B \cdots \otimes_B P_A$ is a separable module. For example, in case ${}_AP \otimes_B P \otimes_B P_A$, instead of above $\boldsymbol{E}$, we may define the map

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$$\boldsymbol{E} : {}_AA_A \longrightarrow {}_AP \otimes_B P \otimes_B P_A \ \ (a \longrightarrow (\boldsymbol{\varepsilon} \otimes \boldsymbol{\varepsilon} \otimes \boldsymbol{\varepsilon})(\sum_{ij} au_iu_j \otimes v_j \otimes v_i)).$$

Example 2. Let A be a commutative ring, A/B a separable extension and ${}_AP_{1A}$ and ${}_AP_{2A}$ be separable modules. We assume $bp_1 = p_1b$ and $bp_2 = p_2b$ for any $p_1 \in P_1$, $p_2 \in P_2$, and $b \in B$. Then ${}_AP_1 \otimes_B P_{2A}$ is a separable module.

Proof. This is similar to Example 1. Let $\boldsymbol{\phi}_i$, $\boldsymbol{\varepsilon}_i$, $\boldsymbol{\pi}_i$ and k_i be maps that satisfy the properties (1), (2), and (3) with respect to ${}_AP_{iA}$ $(i = 1, 2)$. We define maps

$\boldsymbol{\Phi} : {}_AP_1 \otimes_B P_{2A} \longrightarrow {}_AA_A \ \ (p_1 \otimes p_2 \longrightarrow \boldsymbol{\phi}_1(p_1)\boldsymbol{\phi}_2(p_2))$,

$\boldsymbol{E} : {}_AA_A \longrightarrow {}_AP_1 \otimes_B P_{2A} \ \ (a \longrightarrow (\boldsymbol{\varepsilon}_1 \otimes \boldsymbol{\varepsilon}_2)k'(a) = (\boldsymbol{\varepsilon}_1 \otimes \boldsymbol{\varepsilon}_2)(\sum_i au_i \otimes v_i))$,

$\boldsymbol{\Pi} : (P_1 \otimes_B P_2) \otimes_B (P_1 \otimes_B P_2) \longrightarrow P_1 \otimes_B P_2 \ \ (p_1 \otimes p_2 \otimes q_1 \otimes q_2 \longrightarrow \boldsymbol{\pi}_1(p_1 \otimes q_1) \otimes \boldsymbol{\pi}_2(p_2 \otimes q_2))$,

and $\boldsymbol{K} = \xi(k_1 \otimes k_2)$ where ξ is an isomorhpism

$\xi : P_1 \otimes_B P_1 \otimes_B P_2 \otimes_B P_2 \longrightarrow P_1 \otimes_B P_2 \otimes_B P_1 \otimes_B P_2 \ \ (p_1 \otimes q_1 \otimes p_2 \otimes q_2 \longrightarrow p_1 \otimes p_2 \otimes q_1 \otimes q_2)$.

Example 3. Let A be a commutative ring, A/B be a separable extension, and P_A be a separable module that is finitely generated and projective. We assume $ap = pa$ for any $p \in P$ and $a \in A$. Then ${}_AHom_A(P_A, A_A)_A$ is a separable module.

Proof. Let $\boldsymbol{\phi}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\pi}$, and k be maps that satisfy the properties (1), (2), and (3). Then we define maps

$\boldsymbol{\Phi} : {}_AHom_A(P_A, A_A)_A \longrightarrow {}_AA_A \ \ (f \longrightarrow f\boldsymbol{\varepsilon}(1))$,

$\boldsymbol{E} : {}_AA_A \longrightarrow {}_AHom_A(P_A, A_A)_A \ \ (a \longrightarrow a\boldsymbol{\phi} = \boldsymbol{\phi}a)$,

and

$\boldsymbol{\Pi} : Hom_A(P_A, A_A) \otimes_B Hom_A(P_A, A_A) \longrightarrow Hom_A(P_A, A_A) \quad (f \otimes g \longrightarrow f\boldsymbol{\varepsilon}g)$.

Moreover since A/B is separable and P_A is finitely generated and projective, the map

$$\begin{array}{lllll}\boldsymbol{K} : Hom_A(P_A, A_A) & \longrightarrow & Hom_A(P_A, A \otimes_B A_A) & \xrightarrow{(*)} & A \otimes_B Hom_A(P_A, A_A) \\ (f & \longrightarrow & k'f & \longrightarrow & \sum_j x_j \otimes \mu_j \end{array}$$

$\longrightarrow Hom_A(P_A, A_A) \otimes_B Hom_A(P_A, A_A)$

$\longrightarrow \sum_j x_j\boldsymbol{\phi} \otimes \mu_j)$

is defined where the map (*) is an isomorphism. Then for any $p \in P$, since

$\sum_j x_j \otimes \mu_j(p) = k'f(p) = \sum_i f(p)u_i \otimes v_i$, $\quad (\boldsymbol{\Phi E})(a) = a\boldsymbol{\phi\varepsilon}(1) = a$ and $[(\boldsymbol{\Pi K})(f)](p) = [\boldsymbol{\Pi}(\sum_j x_j\boldsymbol{\phi} \otimes \mu_j)](p)$
$= \sum_j x_j\boldsymbol{\phi\varepsilon}\mu_j(p) = \sum_j x_j\mu_j(p) = \sum_i f(p)u_iv_i = f(p)$,
$\boldsymbol{\Phi}$, $\boldsymbol{E}$, $\boldsymbol{\Pi}$, and $\boldsymbol{K}$ satisfy properties (1), (2), and (3).

Example 4. Let B be a commutative ring, A be a separable B-algebra and ${}_AP_A$ be a bimodule such that $bp = pb$ for any $p \in P$ and $b \in B$. If P_A is finitely generated and projective and a generator, then ${}_AHom_A(P_A, A_A) \otimes_B P_A$ is a separable module.

Proof. Since P_A is a generator, there exists $f_i \in Hom_A(P_A, A_A)$ and $p_i \in P$ $(i = 1, \cdots, n)$ such that $\sum_i f_i(p_i) = 1$. For the map $\boldsymbol{\phi} : {}_AHom_A(P_A, A_A) \otimes_B P_A \longrightarrow {}_AA_A \ \ (f \otimes p \longrightarrow f(p))$, we define $\boldsymbol{\varepsilon} : {}_AA_A \longrightarrow {}_AHom_A(P_A, A_A) \otimes_B P_A \ \ (a \longrightarrow \boldsymbol{\rho}k'(a))$ where $\boldsymbol{\rho} : {}_AA \otimes_B A_A \longrightarrow {}_AHom_A(P_A, A_A) \otimes_B P_A$ $(a_1 \otimes a_2 \longrightarrow \sum_i a_1f_i \otimes p_ia_2)$. Then we have $\boldsymbol{\phi\varepsilon}(a) = \boldsymbol{\phi\rho}(\sum_j au_j \otimes v_j) = \boldsymbol{\phi}(\sum_{i,j} au_jf_i \otimes p_iv_j) = \sum_{i,j} au_jf_i(p_i)v_j = a$.

Since P_A is finitely generated and projective, there exists $\boldsymbol{\alpha}_t \in Hom_A(P_A, A_A)$ and $q_t \in P$ $(t = 1, \cdots, l)$ such that $p = \sum_t q_t\boldsymbol{\alpha}_t(p)$ for any $p \in P$.

Let $\boldsymbol{\pi} : (Hom_A(P_A, A_A) \otimes_B P) \otimes_B (Hom_A(P_A, A_A) \otimes_B P) \longrightarrow Hom_A(P_A, A_A) \otimes_B P$
$(f \otimes x \otimes g \otimes y \longrightarrow f \otimes xg(y))$

and

$k : Hom_A(P_A, A_A) \otimes_B P \longrightarrow (Hom_A(P_A, A_A) \otimes_B P) \otimes_B (Hom_A(P_A, A_A) \otimes_B P)$
$(\mu \otimes q \longrightarrow \sum_t \mu \otimes q_t \otimes \boldsymbol{\alpha}_t \otimes q)$.

Then we have $\pi k = id$ and $\phi\pi = \pi'(\phi \otimes \phi)$.

In [1], Sugano called A P-separable over B if $\phi : {}_AHom_A(P_A, A_A) \otimes {}_BP_A \longrightarrow {}_AA_A\ (f \otimes p \longrightarrow f(p))$ splits.

Let A/B be a separable extension. According to [2] Chapter 11, we will note some properties for a separable module ${}_AP_A$.

Let ϕ, ε, π, and k be maps that satisfy the properties (1), (2), and (3). For an A-A-bimodule ${}_AM_A$, let
$V_M(B) = \{m \in M | bm = mb$ for all $b \in B\}$,
$\delta^0{}_P : V_M(B) \longrightarrow Hom({}_BP_B, {}_BM_B)\ (u \longrightarrow (p \longrightarrow \phi(p)u - u\phi(p)))$,
$\delta^1{}_P : Hom({}_BP_B, {}_BM_B) \longrightarrow Hom({}_BP \otimes {}_BP_B, {}_BM_B)\ (\Phi \longrightarrow (p_1 \otimes p_2 \longrightarrow \phi(p_1)\Phi(p_2) - \Phi\pi(p_1 \otimes p_2) + \Phi(p_1)\phi(p_2))$,
$\delta^0{}_A : V_M(B) \longrightarrow Hom({}_BA_B, {}_BM_B)\ (u \longrightarrow (a \longrightarrow au - ua))$, and
$\delta^1{}_A : Hom({}_BA_B, {}_BM_B) \longrightarrow Hom({}_BA \otimes {}_BA_B, {}_BM_B)\ (\Phi' \longrightarrow (a_1 \otimes a_2 \longrightarrow a_1\Phi'(a_2) - \Phi'(a_1a_2) + \Phi'(a_1)a_2))$.

$\Phi' \in Hom(({}_BA_B, {}_BM_B)$ is called a derivation if $\Phi' \in ker\,\delta^1{}_A$, so we call $\Phi \in Hom({}_BP_B, {}_BM_B)$ a derivation if $\Phi \in ker\ \delta^1{}_P$. We can see if $\Phi \in Im\,\delta^0{}_P$, then $ker\phi \subset ker\Phi$ and Φ is a derivation. Furthermore, we have

(1) if $\Phi \in Hom({}_BP_B, {}_BM_B)$ is a derivation and $ker\ \phi \subset ker\ \Phi$, then $\Phi\varepsilon \in Hom({}_BA_B, {}_BM_B)$ is a derivation and $\Phi = \Phi\varepsilon\phi$,

(2) if $\Phi' \in Hom({}_BA_B, {}_BM_B)$ is a derivation then $\Phi'\phi \in Hom({}_BP_B, {}_BM_B)$ is a derivation and $ker\ \phi \subset ker\ \Phi'\phi$,

(3) the map $Hom({}_Aker\ \pi'{}_A, {}_AM_A) \longrightarrow \{\Phi \in Hom({}_BP_B, {}_BM_B) | \Phi$ is a derivation such that $ker\ \phi \subset ker\ \Phi\}$
$\eta' \longrightarrow (p \longrightarrow \eta'(\phi(p) \otimes 1 - 1 \otimes \phi(p)))$
is an isomorphism and the inverse map is given by $(x \otimes y \longrightarrow (\Phi\varepsilon)(x)y) \longleftarrow \Phi$.

References

[1] Kozo Sugano. 1971. Note on Separability of Endomorphism Rings. J. Fac. Sci. Hokkaido Univ. vol. 21. pp. 196-208.

[2] R.S. Pierce. 1982. Associative Algebras. Springer-Verlag. Graduate Texts in Math. 88.

要　約

環の分離拡大に関連して分離加群を定義し，その例について述べた。