J. Coll. Dairying, 7, 1~4 ; (1977)

A remark on automorhisms of separable algebras over Artinian rings

Ryo SAITO*

(May, 1976)

1. Introduction :

Let A be a separable R-algebra, B a separable R-subalgebra of A and σ be any R-algebra monomorphism from B into A. In [2],[3], and [4], the conditions under which the σ can be extended to an inner automorphism of A were studied. The purpose of this note is to prove the following theorem.

Theorem (Theorem (2.2)) Let R be a commutative Artinian ring, A a central separable R-algebra and B an R-subalgebra of A such that ${}_BB\underset{R}{\otimes}B_B$ is isomorphic to ${}_BB_B^{(t)}$ for some integer t. Then if ${}_BA$ is projective, any R-algebra monomorphism σ from B into A can be extended to an inner automorphism of A.

Throughout this note, all rings have an identity 1, all modules over a ring (${}_AM$ will mean a left A-module M) are unitary and a subring of a ring contains the identity of the ring. We consider a ring homomorphism which maps 1 to 1.

We give some definitions and notations before we state its proof.

(1) Semi-local (reap. Local) ring is a commutative ring which has a finite number of (resp. only one) maximal ideals. It is well known that a commutative Artinian ring is a semi-local ring.

(2) Let A be any ring, and ${}_AM$ and ${}_AN$ be any left A-modules. We write ${}_AM|{}_AN$ if ${}_AM$ is isomorphic to a direct summand of a direct sum of a finite number of copies of ${}_AN$.

(3) ${}_AN^{(t)}$ is the direct sum of t-copies of ${}_AN$.

(4) We call a ring A a quasi-Frobenius ring when A is a left Artinian ring and ${}_AA$ is an injective left A-module. ([6]p.12)

(5) Let A be an R-algebra, B a subalgebra of A and σ be any R-algebra monomorphism from B into A. A can be regarded canonically as a left B-module. Moreover A can be regarded as a new left B-module by $b*a=\sigma(b)a$. This new left B-module is written as ${}_{B\sigma}A$. It is well known that σ can be extended to an inner automorphism

* Lab. of Mathematics, The College of Dairying, Ebetsu, Hokkaido, Japan

of A if and only if A is isomorphic to ${}_\sigma A$ as a left $B\underset{R}{\otimes}A^\circ$-module, where A° is an opposite ring of A. ([7]). Here we define the product of an element $b\otimes a^\circ$ of $B\underset{R}{\otimes}A^\circ$ and an ement x of A as follows.

$$(b\otimes a^\circ)x=bxa \qquad \text{in } A$$
$$(b\otimes a^\circ)*x=\sigma(b)xa \qquad \text{in } {}_\sigma A$$

2. Proof of the theorem

Lemma (2.1) Let R be a commutative Artinian ring, A an R-algebra which is a finitely generated R-module and B be an R-subalgebra of A. Then if there exists an R-algebra D which contains B and an integer t such that $D\underset{R}{\otimes}B$ is isomorphic to $B^{(t)}$ as a B-B-bimodule, A is isomorphic to ${}_\sigma A$ as a left B-module for any R-algebra monomorphism σ from B into A.

Proof. As A is isomorphic to ${}_\sigma A$ as an R-module, $D\underset{R}{\otimes}A$ is isomorphic to $D\underset{R}{\otimes}{}_\sigma A$ as a left D-module. Moreover the following diagram is commutative and the all arrows are left D-isomorphisms

$$\begin{tikzcd}
D\underset{R}{\otimes}A \arrow[r] & D\underset{R}{\otimes}{}_\sigma A \arrow[d] \\
D\underset{R}{\otimes}B\underset{B}{\otimes}A \arrow[u] \arrow[r] & D\underset{R}{\otimes}B\underset{B}{\otimes}{}_\sigma A
\end{tikzcd}
\qquad
\begin{tikzcd}
d\otimes ba \arrow[r] & d\otimes ba \arrow[d] \\
d\otimes b\otimes a \arrow[u] \arrow[r] & d\otimes 1\otimes ba
\end{tikzcd}$$

Since ${}_B D\underset{R}{\otimes}B_B$ is isomorphic to ${}_B B_B^{(t)}$, ${}_B A^{(t)}$ is isomorphic to ${}_{B\sigma}A^{(t)}$.

Since B is an Artinian ring and ${}_B A$ is finitely generated, ${}_B A$ and ${}_{B\sigma}A$ are the direct sum of a finite number of indecomposable left B-modules.
If we put

$${}_B A=U_1\oplus\cdots\cdots\cdots\oplus U_n$$
$${}_{B\sigma}A=V_1\oplus\cdots\cdots\cdots\oplus V_m,$$

$(U_1\oplus\cdots\oplus U_n)^{(t)}$ is isomorphic to $(V_1\oplus\cdots\oplus V_m)^{(t)}$. By Krull-Remak-Schmidt-Azumaya's theorem, n equals m and U_i is isomorphic to $V_{\pi(i)}$ where π is a permutation of $\{1,2,\cdots\cdots,n\}$. Hence ${}_B A$ is isomorphic to ${}_{B\sigma}A$. Q.E.D.

Theorem (2.2) Let R be a commutative Artinian ring, A a central separable R-algebra and B be R-subalgebra of A. Then if there exists an integer t such that ${}_B B\underset{R}{\otimes}B_B$ is isomorphic to ${}_B B_B^{(t)}$ as a B-B-bimodule and if ${}_B A$ is projective, any R-algebra monomorphism σ from B into A can be extended to an inner automorphism of A.

Proof. By Lemma (2.1), ${}_B A$ is isomorphic to ${}_{B\sigma}A$ and ${}_{B\sigma}A$ is projective. Moreover ${}_C A$ is isomorphic to ${}_{C\sigma}A$ where C is the center of B. Since A is a separable R-algebra, ${}_{A\underset{R}{\otimes}A^\circ}A\,|\,{}_{A\underset{R}{\otimes}A^\circ}A\underset{R}{\otimes}A^\circ$. Hence ${}_{B\underset{R}{\otimes}A^\circ}A\,|\,{}_{B\underset{R}{\otimes}A^\circ}A\underset{R}{\otimes}A^\circ$ and ${}_{B\underset{R}{\otimes}A^\circ\sigma}A\,|\,{}_{B\underset{R}{\otimes}A^\circ\sigma}A\underset{R}{\otimes}A^\circ$. On the other hand, ${}_{B}A|_{B}B$ and ${}_{B\sigma}A|_{B}B$ so ${}_{B\otimes_R A^\circ}A|_{B\otimes_R A^\circ}A\otimes_R A^\circ|_{B\otimes_R A^\circ}B\otimes_R A^\circ$ and ${}_{B\otimes_R A^\circ\sigma}A|_{B\otimes_R A^\circ\sigma}A\otimes_R A^\circ|_{B\otimes_R A^\circ}B\otimes_R A^\circ$. Therefore ${}_{B\otimes_R A^\circ}A$ and ${}_{B\otimes_R A^\circ\sigma}A$ are projective. By [8] Theorem 2.2, B is a separable R-algebra and hence $B\otimes_R A^\circ$ is a central separable algebra over semil-local ring C. Thus by [2] Theorem 1.1, ${}_{B\otimes_R A^\circ}A$ is isomorphic to ${}_{B\otimes_R A^\circ\sigma}A$. Hence σ can be extended to an inner automorphism of A. Q.E.D.

Remark. In [1], Azumaya called an R-algebra B which satisfies the condition that ${}_{B\otimes_R A^\circ}A$ is projective relatively separable in A.

Lemma (2.3) Let S be a commutative quasi-Frobenius ring and M be a faithful S-module. Then S is isomorphic to a direct summand of M as an S-module.

Proof. In the case where S is a local ring, the socle S_o of S is simple. Since M is faithful, $S_oM\neq 0$. Hence there exists an element $m_o\in M$ such that $S_om_o\neq 0$. We can define an S-module homomorphism μ from S into $M(\mu(s)=sm_o,\ s\in S)$ and the kernel of μ equals 0. Since S is an injective S-module, S is isomorphic to a direct summand of M as an S-module. Next when S is an arbitrary commutative Artinian ring, we can decompose S into a direct sum of local rings. Let $S=S_1\oplus\cdots\oplus S_n$ be such a decomposition. Since S is a quasi-Frobeius ring, each S_i is a quasi-Frobenius ring. In this case, M is equal to $S_1M\oplus\cdots\oplus S_nM$ and each S_iM is a faithful S_i-module. By the first part of of this lemma, each S_iM is a direct sum of U_i and V_i where U_i is isomorphic to S_i as an S_i-module. Then we have

$$ {}_SM=\bigoplus_{i=1}^{n} S_iM=\bigoplus_{i=1}^{n}(U_i\oplus V_i)\cong\left(\bigoplus_{i=1}^{n}U_i\right)\oplus\left(\bigoplus_{i=1}^{n}V_i\right)\cong{}_SS\oplus\left(\bigoplus_{i=1}^{n}V_i\right). \quad \text{Q.E.D.} $$

Proposition (2.4) (c.f. (9] Theorem3) Let S be a commutative R-algebra which is a finitely generated projective R-module. If $A=Hom_R(S,S)$ and S is a quasi-Frobeninius ring, the following conditions are satisfied.

(1) $V_A(S)=S$ where $V_A(S)=\{a\in A|sa=as$ for all $s\in S\}$.

(2) Every R-algebra monomorphism from S into A can be extended to an inner automorphism of A.

Proof. (1) $V_A(S)=V_{Hom_R(S,S)}(S)=Hom_S(S,S)=S$.

(2) Let σ be an arbitrary R-algebra monomorphism from S into A. Then S has two left S-module structures. The first is an ordinary one and the second is defined by $s_*x=\sigma(s)x$ for $s\in S$ and $x\in S$. We denote the second left S-module S by ${}_\sigma S$. ${}_\sigma S$ is a faithful left S-module. Hence by lemma (2.3), S is isomorphic to a direct summand of ${}_\sigma S$ as a left S-module and also as an R-module. ${}_\sigma S\cong{}_SS\oplus_SX$. For any maximal ideal $\mathfrak{m}$ of R, we have $R_{\mathfrak{m}}\otimes_{R\sigma}S\cong(R_{\mathfrak{m}}\otimes_RS)\oplus(R_{\mathfrak{m}}\otimes_RX)$ and rank ${}_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}\otimes_{R_\sigma}S)=$ rank ${}_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}\otimes_RS)$ where $R_{\mathfrak{m}}$ is the localization at $\mathfrak{m}$. Hence $R_{\mathfrak{m}}\otimes_RX=0$ and $X=0$; that is S is isomorphic to ${}_\sigma S$ as a left S-module. We put this isomorphism from ${}_SS$ to ${}_{S\sigma}S$ a. Then a is an invertible element of

$A = \mathrm{Hom}_R(S,S)$ and for any $s,x \in S$, $(as)(x) = a(sx) = s_*(ax) = (\sigma(s)a)(x)$. Hence $as = \sigma(s)a$. That is $\sigma(s) = asa^{-1}$. Q.E.D.

Remark. It is known that commutative QF-1 Artinian rings are quasi-Frobenius rings. ([5]).

The present author wishes to express his heartfelt thanks to Prof. T. Onodera of Hokkaido University for his constructive advice.

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