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A note on automorphisms of separable closures of commutative rings

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§1. Introduction

Throughout this note, all rings are commutative and have an identity element 1. Let R be a ring, Ω be a separable closure of R, and G be the group of all R-algebra automorphisms of Ω. Then a continuous function $f: X(\Omega)\to X(R)$ $(x=C(\mathfrak{p})\to f(x)=C(\mathfrak{p}\cap R))$ is defined. Let $R_{f(x)}=R/I_R(f(x))$, $\Omega_{f(x)}=\Omega\otimes_R R_{f(x)}$ and $\Omega_x=\Omega/I_\Omega(x)$ (Definitions are found in §2.). We can define two canonical epimorphisms as follows.

$$g: \Omega_{f(x)}\to \Omega_x \ (\omega\otimes r_{f(x)}\to(\omega r)_x)$$
$$u: \Omega\to\Omega_x(\omega\to\omega_x)$$

As Ω is a separable closure of R, $\Omega_{f(x)}$ is a locally strongly separable extension of $R_{f(x)}$. We set $u(R)=g(R_{f(x)})=R(x)$; that is $R(x)=\{r_x\in\Omega_x | r\in R\}$. Then Ω_x is a separable closure of $R(x)$ (Proposition 1). Let $G(x)$ be the group of all $R(x)$-algebra automorphisms of Ω_x and $G_\mathfrak{p}=\{\sigma\in G|$ for any idempotent $e\in\Omega$, if $e\in\mathfrak{p}$, then $\sigma(e)\in\mathfrak{p}\}$. Then $G_\mathfrak{p}$ is a subgroup of G (Lemma 2), and any $\sigma\in G_\mathfrak{p}$ induces an automorphism $\bar{\sigma}$ of Ω_x (Lemma 3). Let $\bar{G}_\mathfrak{p}=\{\bar{\sigma}|\sigma\in G_\mathfrak{p}\}\subset G(x)$. The main theme of this note is to investigate the structure of $G_\mathfrak{p}$, $\bar{G}_\mathfrak{p}$, G and $G(x)$.

§2. Definitions and notations

In this section, insofar as required, we describe definitions and notations which are found in [1] or [3].

(1) A ring is called connected if it has no idempotents except 0 and 1.

(2) A ring extension S/R is called a strongly separable extension if S is a separable R-algebra and a finitely generated projective R-module. A ring extension S/R is called a locally strongly separable extension if S is the direct limit of strongly separable extensions of R.

(3) Let $X(R)$ be the space of the connected components of Spec (R). Then $X(R)$ is a profinite topological space ([3] p. 26). For $x=C(\mathfrak{p})\in X(R)$, we set $I_R(x)$ = the ideal genrated by idempotents of $\mathfrak{p}$, where $C(\mathfrak{p})$ is the connected component

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of $\mathfrak{p} \in \operatorname{Spec}(R)$. $I_R(x)$ takes the form of

$$I_R(x)=\{re \mid e \text{ is an idempotent of } \mathfrak{p}, r \in R\}.$$

Further we set $R_x = R/I_R(x)$. R_x is a connected ring and a flat R-module ([3] pp. 33-34). For an R-module M, we set $M_x = M \otimes_R R_x = M/I_R(x)M$. For any $m \in M$, we identify m_x, $m \otimes 1_x$ and $\bar{m}$.

(4) For a ring extension S/R, S is called a componentially locally strongly separable extension if for any $x \in X(R)$, S_x is a locally strongly separable extension of R_x ([3] p. 79).

(5) Let R be a connected ring. R is called separably closed ring if its only connected strongly separable extension is itself. For a connected ring R, there exists uniquely up to isomorphisms a ring extension of R which has the following properties. Its ring extension is

(a) connected,
(b) a locally strongly separable extension of R,
(c) and separably closed.

This ring extension is called a separable closure of R, and is denoted by Ω ([1] p. 103 and [3] p. 51).

(6) Let S be a ring (not necessarily connected). S is called separably closed if for any componentially locally strongly separable extension T of S, there exists an S-algebra homomorphism from T to S. If S is separably closed, then for any $x \in X(S)$, S_x is separably closed in the sense of (5) ([3] p. 90, 91). Let S be a componentially locally strongly separable extension of R. S is called minimal if for any componentially locally strongly separable extension T of R, every R-algebra homomorphism from S to T is a monomorphism ([3] p. 92). For any ring R, there exists uniquely up to isomorphisms a ring extension of R which has the following properties. Its ring extension is

(a) minimal,
(b) a componentially locally strongly separable extension of R,
(c) and separably closed.

This ring extension is called a separable closure of R, and is dented by Ω ([3] p. 93).

§ 3. Results

Proposition 1. Ω_x is a separable closure of $R(x)$.

Proof By §2 (3) and (6), Ω_x is connected and separably closed. As $I_\Omega(x)$ is generated by idemptents of Ω, Ker $g=\{\omega_{f(x)} \in \Omega_{f(x)} \mid \omega \in I_\Omega(x)\}$ is generated by idempotents of $\Omega_{f(x)}$. Hence by [3] p. 48 Proposition 3.5, Ω_x is a locally strongly separable extension of $R(x)$. Thus Ω_x is a separable closure of $R(x)$ (See §2(5)).

Lemma 2. $G_\mathfrak{p}$ is a subgroup of G, and if $C(\mathfrak{p})=C(\mathfrak{q})$, then $G_\mathfrak{p}=G_\mathfrak{q}$.

Proof. First we prove that

$$G_{\mathfrak{p}}=\{\sigma\in G \mid \text{for any idempotent } e\in\Omega, \text{ if } \sigma(e)\in\mathfrak{p},\ e\in\mathfrak{p}\}.$$

Assume that $\sigma\in G_{\mathfrak{p}}$ and $\sigma(e)\in\mathfrak{p}$. If $e\notin\mathfrak{p}$, then $1-e\in\mathfrak{p}$ hence $\sigma(1-e)=1-\sigma(e)\in\mathfrak{p}$. Thus, $1=1-\sigma(e)+\sigma(e)\in\mathfrak{p}$. This is a contradiction. Hence $e\in\mathfrak{p}$. Conversely, we assume for $\sigma\in G$ that for any idempotent $e\in\Omega$, if $\sigma(e)\in\mathfrak{p}$, then $e\in\mathfrak{p}$. Let $e\in\mathfrak{p}$ and $\sigma(e)\notin\mathfrak{p}$. $\mathfrak{p}\ni 1-\sigma(e)=\sigma(1-e)$. Thus, $1-e\in\mathfrak{p}$. This is a contradiction. Hence $\sigma(e)\in\mathfrak{p}$. Obviously $G_{\mathfrak{p}}$ is closed under the multiplication. Let $\sigma\in G_{\mathfrak{p}}$. For any idempotent $e\in\mathfrak{p}$, $\sigma(\sigma^{-1}(e))=e\in\mathfrak{p}$. Hence by the first part $\sigma^{-1}(e)\in\mathfrak{p}$. So $\sigma^{-1}\in G_{\mathfrak{p}}$. Consequently $G_{\mathfrak{p}}$ is a subgroup of G. When $C(\mathfrak{p})=C(\mathfrak{q})$, the set of idempotents of $\mathfrak{p}$ equals the set of idempotents of $\mathfrak{q}$ ([3] p. 26). Hence $G_{\mathfrak{p}}=G_{\mathfrak{q}}$.

Lemma 3. Every element $\sigma\in G_{\mathfrak{p}}$ induces an $R(x)$-algebra automorphism of Ω_x. This induced $R(x)$-algebra automorphism is written by $\bar{\sigma}$.

Proof. Obviously $\bar{\sigma}(\omega_x)=\sigma(\omega)_x$ is well defined.

$$\begin{tikzcd} \Omega \arrow[r, "\sigma"] \arrow[d, "u"'] & \Omega \arrow[d, "u"] \\ \Omega_x \arrow[r, "\bar{\sigma}"] & \Omega_x \end{tikzcd} \qquad \begin{tikzcd} \omega \arrow[r] \arrow[d] & \sigma(\omega) \arrow[d] \\ \omega_x \arrow[r] & \sigma(\omega)_x=\bar{\sigma}(\omega_x) \end{tikzcd}$$

We assume that $\sigma(\omega)_x=0$. Then $\sigma(\omega)\in I_\Omega(x)$ and $\sigma(\omega)=\sigma(\omega)e'$ for some idempotent $e'\in\mathfrak{p}$. So $\omega=\sigma^{-1}(\sigma(\omega))=\sigma^{-1}(\sigma(\omega)e')=\omega\sigma^{-1}(e')\in I_\Omega(x)$ and $\omega_x=0$. Consequently $\bar{\sigma}$ is a monomorphism. For any $\omega'_x\in\Omega_x$, we choose ω as $\sigma(\omega)=\omega'$. Then we have $\bar{\sigma}(\omega_x)=\sigma(\omega)_x=\omega'_x$ and hence $\bar{\sigma}$ is an epimorphism.

Let $\bar{G}_{\mathfrak{p}}=\{\bar{\sigma}\mid\sigma\in G_{\mathfrak{p}}\}$ and $G(x)$ be the full group of $R(x)$-algebra automorphisms of Ω_x.

Proposition 4.

(1) For any $\sigma\in G$, σ induces an $R(x)$-algebra automorphism of Ω_x if and only if $\sigma\in G_{\mathfrak{p}}$.

(2) For any $\alpha\in G(x)$, α is induced by an element of G if and only if $\alpha\in\bar{G}_{\mathfrak{p}}$.

Proof. (1). We assume that σ induces an $R(x)$-algebra automorphism $\bar{\sigma}$ of Ω_x. Then the following diagram is commutative.

$$\begin{tikzcd} \Omega \arrow[r, "\sigma"] \arrow[d, "u"'] & \Omega \arrow[d, "u"] \\ \Omega_x \arrow[r, "\bar{\sigma}"] & \Omega_x \end{tikzcd}$$

For any idempotent $e\in\mathfrak{p}$, $0=\bar{\sigma}(e_x)=\sigma(e)_x$. Hence $\sigma(e)\in I_\Omega(x)\subset\mathfrak{p}$ and $\sigma\in G_{\mathfrak{p}}$. The converse holds by Lemma 3.

(2). This is trivial.

By [1] p. 107, we know that $G(x)=\varprojlim G(S)$ where S is a strongly separable extension of $R(x)$ in Ω_x, and $G(S)$ is the set of all $R(x)$-algebra homomorphisms from S to Ω_x. $G(S)$ is a finite set. For any strongly separable extension S of $R(x)$ in Ω_x, we set

$$u^{-1}(S)=\{\omega\in\Omega \mid u(\omega)\in S\}$$

and

$$G(u^{-1}(S))=\left\{p \,\middle|\, \begin{array}{l} p \text{ is an } R\text{-algebra monomorphism from } u^{-1}(S) \text{ to } \Omega \\ \text{such that } p(I_\Omega(x))=I_\Omega(x). \end{array}\right\}$$

Each $u^{-1}(S)$ is an R-subalgebra of Ω and $\{u^{-1}(S)\}$ is a directed set under inclusion.

Theorem 5.

(1) $\Omega=\varinjlim u^{-1}(S)$

(2) $\Omega_x=\varinjlim S$

(3) Any element of $G(u^{-1}(S))$ induces an element of $G(S)$.

(4) $G_\mathfrak{p}=\varprojlim G(u^{-1}(S))$

Proof. (1), (2) and (3) are trivial.

(4) We have

$$\varprojlim G(u^{-1}(S))=\left\{ (\sigma_{u^{-1}(S)})\in\prod G(u^{-1}(S)) \,\middle|\, \begin{array}{l} \text{for any strongly separable} \\ \text{extensions } S_1 \text{ and } S_2 \text{ such} \\ \text{that } S_1\subset S_2,\ \sigma_{u^{-1}(S_2)} \mid u^{-1}(S_1) \\ \text{(restriction map)} = \sigma_{u^{-1}(S_1)}. \end{array}\right\}$$

For any $\sigma\in G_\mathfrak{p}$, obviously $(\sigma|u^{-1}(S))\in\varprojlim G(u^{-1}(S))$. Conversely for any $(\sigma_{u^{-1}(S)})\in\varprojlim G(u^{-1}(S))$, we show the map

$$\sigma:\Omega\to\Omega \quad (\omega\to\sigma_{u^{-1}(S)}(\omega)) \ (\omega_x\in S)$$

belongs to $G_\mathfrak{p}$. Let $\omega_x\in S_1$ and $\omega_x\in S_2$ for strongly separable extensions S_1 and S_2. Then there exists a strongly separable extension S such that $S_1\subset S$ and $S_2\subset S$, and $\sigma_{u^{-1}(S_1)}(\omega)=\sigma_{u^{-1}(S)}(\omega)=\sigma_{u^{-1}(S_2)}(\omega)$. Hence σ is independent of the choice of S. Since each $\sigma_{u^{-1}(S)}$ is a monomorphism, σ is a monomorphism. Let (α_S) be the element of $G(x)$ such that each α_S is induced by $\sigma_{u^{-1}(S)}$. For any $\omega'\in\Omega$, there exists $\omega\in\Omega$ such that $((\alpha_S))(\omega_x)=\omega'_x$. We choose a strongly separable extension S which satisfies $\omega_x\in S$. Then $\sigma_{u^{-1}(S)}(\omega)_x=\alpha_S(\omega_x)=\omega'_x$; that is $\omega'-\sigma_{u^{-1}(S)}(\omega)\in I_\Omega(x)$. Hence there exists $t\in I_\Omega(x)$ such that $\omega'-\sigma_{u^{-1}(S)}(\omega)=\sigma_{u^{-1}(S)}(t)$; that is $\omega'=\sigma_{u^{-1}(S)}(\omega+t)$, and σ is an epimorphism. For any idempotent $e\in\mathfrak{p}$, $e_x=0$. Hence for any strongly separable extension S, $\sigma(e)_x=\sigma_{u^{-1}(S)}(e)_x=\alpha_S(e_x)=0$ and $\sigma_{u^{-1}(S)}(e)\in I_\Omega(x)\subset\mathfrak{p}$. Consequently $\sigma\in G_\mathfrak{p}$, and

$$\xi: G_\mathfrak{p}\to\varprojlim G(u^{-1}(S)) \quad (\sigma\to(\sigma|u^{-1}(S)))$$

and

$$\eta:\varprojlim G(u^{-1}(S))\to G_\mathfrak{p} \quad ((\sigma_{u^{-1}(S)})\to(\omega\to\sigma_{u^{-1}(S)}(\omega)))$$

are inverse bijections each other.

$G(x)=\varprojlim G(S)$ (resp. $G_\mathfrak{p}=\varprojlim G(u^{-1}(S))$) is a topological space under the standard topology (cf. [3] p. 58) ; that is, $G(x)$ (resp. $G_{\mathfrak{p}}$) has the inverse limit topology induced from the discreate toplogy on each $G(S)$ (resp. $G(u^{-1}(S))$). On the other hand, $G(x)$ (resp. G) has the finite topology. The element of the subbasis for the neighborhoods of α(resp. σ) takes the form of $\{\beta\in G(x)|\alpha(\omega_x)=\beta(\omega_x)\}$(resp. $\{\tau\in G|\sigma(\omega)=\tau(\omega)\}$) ([2] p. 29).

Lemma 6. $G(x)$ (resp. G) is a topological group under the finite topology.

Proof. $G(x)$ (resp. G) is a subgroup of the automorphism group A of Ω_x (resp. Ω) as an Abelian group. Since A is a topological group under the finite topolgy ([2] p. 30), $G(x)$ (resp. G) is a topological group under the finite topology.

Proposition 7. In $G(x)$, the finite topology is equivalent to the standard topology.

Proof. The fundamental neighborhood system of $\alpha=(\alpha_S)$ in $G(x)$ under the standard topology takes the form of

$$U=(\alpha_{S_1}\times\cdots\times\alpha_{S_n}\times\prod_{S\neq S_1,S_2,\cdots,S_n}G(S))\cap G(x).$$

Let $\{s_{ij}|j=1,2,\cdots,m_i\}$ be the set of generators of the strongly separable extension S_i $(i=1,2,\cdots,n)$ as an $R(x)$-module. Then we have

$$U=\bigcap_{\substack{s_{ij}\in S_i\\ i=1,2,\cdots,n\\ j=1,2,\cdots,m_i}}\{\beta\in G(x)|\beta(s_{ij})=\alpha(s_{ij})\}.$$

Hence the finite topology is stronger than the standard topology. Conversely, let $V=\{\beta\in G(x)|\beta(s)=\alpha(s)\}$ be any element of the subbasis of neighborhood of α and $s\in S_1$ for some strongly separable extension.

For any $\beta\in V$

$$V=\{\beta\in G(x)|\beta(s)=\alpha(s)\}\supset(\beta_{S_1}\times\prod_{S\neq S_1}G(S))\cap G(x)\ni\beta$$

Hence V is an open set in $G(x)$ under the standard topology and the standard topology is stronger than the finite topology.

Corollary 8. (cf. [1] p.107) $G(x)$ is a topological group under the standard topology.

Proposition 9. $G_{\mathfrak{p}}$ is a closed topological subgroup of G under the finite topology.

Proof. We will show that $G_{\mathfrak{p}}^C$ (complement in G) is open. Assume that $\sigma\in G_{\mathfrak{p}}^C$. Then there exists an idempotent $e\in\mathfrak{p}$ such that $\sigma(e)\notin\mathfrak{p}$. $\{\tau\in G|\tau(e)=\sigma(e)\}$ is an open set in G which contains σ, and we have $\{\tau\in G|\tau(e)=\sigma(e)\}\cap G_{\mathfrak{p}}=\emptyset$. Thus $G_{\mathfrak{p}}$ is closed in G.

Proposition 10. In $G_{\mathfrak{p}}$, the standard topology is stronger than the finite topology.

Proof. Using Theorem 5(1) and (4), we can prove in a similar manner to the second part of Proposition 7.

Proposition 11. $G_\mathfrak{p}$ is a topological group under the standard topolgy.

Proof. We will show that the maps $G_\mathfrak{p}\times G_\mathfrak{p}\to G_\mathfrak{p}((\sigma,\tau)\to\sigma\tau)$ and $G_\mathfrak{p}\to G_\mathfrak{p}(\sigma\to\sigma^{-1})$ are continuous. For any element of the subbasis of the neighborhoods $(\sigma\tau|u^{-1}(S_1)\times\prod_{S\neq S_1}G(u^{-1}(S)))\cap G_\mathfrak{p}$ of $\sigma\tau$, we take the neighborhood
$(\tau|u^{-1}(S_1)\times\prod_{S\neq S_1}G(u^{-1}(S)))\cap G_\mathfrak{p}$ of τ and the neighborhood
$(\sigma|u^{-1}(S_2)\times\prod_{S\neq S_2}G(u^{-1}(S)))\cap G_\mathfrak{p}$ of σ.
Here $S_2=\bar{\tau}(S_1)$.
Since $\tau(u^{-1}(S_1))\subset u^{-1}(\bar{\tau}(S_1))=u^{-1}(S_2)$, we have

$$((\sigma|u^{-1}(S_2)\times\prod_{S\neq S_2}G(u^{-1}(S)))\cap G_\mathfrak{p})((\tau|u^{-1}(S_1)\times\prod_{S\neq S_1}G(u^{-1}(S)))\cap G_\mathfrak{p})$$
$$\subset(\sigma\tau|u^{-1}(S_1)\times\prod_{S\neq S_1}G(u^{-1}(S)))\cap G_\mathfrak{p}$$

Hence the map $G_\mathfrak{p}\times G_\mathfrak{p}\to G_\mathfrak{p}((\sigma,\tau)\to\sigma\tau)$ is continuous. For any element of the subbasis of the neighborhoods $(\sigma^{-1}|u^{-1}(S_1)\times\prod_{S\neq S_1}G(u^{-1}(S)))\cap G_\mathfrak{p}$ of σ^{-1}, let $\overline{\sigma^{-1}}(S_1)=S_2$. Since $\sigma^{-1}(u^{-1}(S_1))\subset u^{-1}(\overline{\sigma^{-1}}(S_1))=u^{-1}(S_2)$, we have

$$\left[\left(\sigma|u^{-1}(S_2)\times\prod_{S\neq S_2}G(u^{-1}(S))\right)\cap G_\mathfrak{p}\right]^{-1}\subset\left(\sigma^{-1}|u^{-1}(S_1)\times\prod_{S\neq S_1}G(u^{-1}(S))\right)\cap G_\mathfrak{p}$$

Hence the map $G_\mathfrak{p}\to G_\mathfrak{p}(\sigma\to\sigma^{-1})$ is continuous.

From the above considerations, we have the following Theorem.

Theorem 12. Let $G_\mathfrak{p}[s]$ and $G_\mathfrak{p}[f]$ be topological groups under the standard topology and the finite topolgy respectively. Then the following arrows are continuous.

$$G_\mathfrak{p}[s]\xrightarrow{1_{G_\mathfrak{p}}}G_\mathfrak{p}[f]\xrightarrow{-}G(x)$$

Proof. $1_{G_\mathfrak{p}}$ is continuous by Proposition 10. For any element of the subbasis of the neighborhoods $V=\{\alpha\in G(x)|\alpha(\omega_x)=\bar{\sigma}(\omega_x)\}$ of $\bar{\sigma}$, $U=\{\tau\in G_\mathfrak{p}|\tau(\omega)=\sigma(\omega)\}$ is a neighborhood of σ. And for any $\tau\in U$, we have $\bar{\tau}\in V$. Hence $-$ is continuous.

Corollary 13. If $G(u^{-1}(S))$ is a finite set for each S, $\bar{G}_\mathfrak{p}$ is a closed subgroup of $G(x)$.

Proof. By the assumption $G(u^{-1}(S))$ is compact and hence $G_\mathfrak{p}=\varprojlim G(u^{-1}(S))$ is compact. Hence $\bar{G}_\mathfrak{p}$ is compact and closed $G(x)$ (cf. [1] p. 107).

Proposition 14. If $I_\Omega(x)$ is a direct summand of Ω, every $R(x)$-algebra monomorphism σ from $u^{-1}(S)$ to Ω which fixes the elements of $I_\Omega(x)$ is extended to an automorphism of Ω.

Proof. By the assumption that $\Omega=I_\Omega(x)\oplus L$ (as Ω-module), we have $u^{-1}(S)=I_\Omega(x)\oplus(L\cap u^{-1}(S))$(as $u^{-1}(S)$-module) and $S\cong u^{-1}(S)\ /I_\Omega(x)\cong L\cap u^{-1}(S)$ (as $R(x)$-algebra). Let $1=e+f$ $(e\in I_\Omega(x), f\in L\cap u^{-1}(S))$. Since $e+f=1=\sigma(1)=\sigma(e)+\sigma(f)=e+\sigma(f)$, $\sigma(f)=f$. For any $k\in L\cap u^{-1}(S)$, since $k=kf$, $\sigma(k)=\sigma(k)\sigma(f)=\sigma(k)f\in L$. Hence $\sigma(L\cap u^{-1}(S))\subset L$. The $R(x)$-algebra homomorphism $\bar{\sigma}$ from S to Ω_x which is induced by σ is extended to an automorphism $\tilde{\sigma}$ of Ω_x ([1] p. 106).

$$\begin{tikzcd}
L \arrow[r, tail, two heads, "p"] & \Omega/I_\Omega(x) \arrow[r, tail, two heads, "q"] & \Omega_x \arrow[r, tail, two heads, "\tilde{\sigma}"] & \Omega_x \arrow[r, tail, two heads, "q^{-1}"] & \Omega/I_\Omega(x) \arrow[r, tail, two heads, "p^{-1}"] & L \\
L\cap u^{-1}(S) \arrow[u, "i"] \arrow[r, tail, two heads] \arrow[urrrrr, bend right, "\sigma|L\cap u^{-1}(S)"'] & u^{-1}(S)/I_\Omega(x) \arrow[u, "i"] \arrow[r, tail, two heads] & S \arrow[u, "i"] \arrow[ur, "\bar{\sigma}"'] & & &
\end{tikzcd}$$

Let $\sigma^*=p^{-1}\circ q^{-1}\circ\tilde{\sigma}\circ q\circ p$. Then the map $\Omega\to\Omega$ $(a+b\to a+\sigma^*(b))$ $(a\in I_\Omega(x), b\in L)$ is an R-algebra automorphism and is an extention of σ.

Proposition 15. Let Ω be a separable closure of R and h be an R-algebra endomorphism of Ω such that for some $x\in X(\Omega)$, $h(I_\Omega(x))=I_\Omega(x)$. Then h is an automorphism.

Proof. The induced $R(x)$-algebra endomorphism $\bar{h}$ of Ω_x is an automorphism ([1] p. 106). For any $\omega'\in\Omega$, there exists $\omega\in\Omega$ such that $h(\omega)_x=\bar{h}(\omega_x)=\omega'_x$. Hence $\omega'-h(\omega)\in I_\Omega(x)$. By the assumption, there exists $t\in I_\Omega(x)$ such that $\omega'-h(\omega)=h(t)$. Hence $h(\omega+t)=\omega'$; that is, h is an epimorphism. By § 2 (6), h is a monomorphism (Ω is minimal). Consequently h is an automorphism.

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