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A Note on Normal Algebras over Commutative Rings

Dedicated to Professor Goro Azumaya on his Sixtieth Birthday

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A finite dimensional field extension L/K is said to be normal if for any field extension P/L and for any K-monomorphism, $f: L \to P$, $f(L)=L$ ([2]; p. 42). Then the following theorem is well known.

Theorem ([2]; p. 52). Let Σ/K be a finite dimensional Galois extension and Γ/K be a finite dimensional purely inseparable extension. Then $\Sigma \otimes_K \Gamma$ is a field and a normal extension of K.

The purpose of this short note is to prove the following theorem which has some relationship to the above theorem.

Theorem. Let R be a commutative ring, S be a commutative G-Galois extension of R ([1]; p. 84) and T be a commutative R-algebra such that $T \otimes_R T \cong T\,(t_1 \otimes t_2 \to t_1 t_2)$. Then $S \otimes_R T$ is a normal R-algebra.

Throught this note, we assume that every ring is commutative with identity 1, every module is unitary, every ring extension has the common 1 and for every ring homomorphism, the image of 1 is 1.

Lemma ([6]; p. 47). Let T be an R-algebra such that $T \otimes_R T \cong T$ $(t_1 \otimes t_2 \to t_1 t_2)$. Then for any T-modules N and M, $\mathrm{Hom}_T(N, M) = \mathrm{Hom}_R(N, M)$.

Proof. It is obvious that $\mathrm{Hom}_T(N, M) \subset \mathrm{Hom}_R(N, M)$. Since ${}_T T \otimes_R N \cong {}_T T \otimes_R (T \otimes_T N) \cong {}_T (T \otimes_R T) \otimes_T N \cong {}_T T \otimes_T N \cong {}_T N$, we have

$$
\begin{array}{ll}
\mathrm{Hom}_T(N, M) & f \\
\cong \mathrm{Hom}_T(T \otimes_R N, M) & \longrightarrow \big(t \otimes n \longrightarrow f(tn) = tf(n)\big) \\
\cong \mathrm{Hom}_R\big(N, \mathrm{Hom}_T(T, M)\big) & \longrightarrow \big(n \longrightarrow \big(t \longrightarrow tf(n)\big)\big) \\
\cong \mathrm{Hom}_R(N, M) & \longrightarrow \big(n \longrightarrow f(n)\big) = f
\end{array}
$$

Hence $\mathrm{Hom}_T(N, M) = \mathrm{Hom}_R(N, M)$.

Definition ([4]; p. 31). Let $X(R)$ be the Boolean spectrum of R and

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$x \in X(R)$. Let $I(x)$ be the ideal generated by the set of all idempotents in any prime ideal belonging to x, $R_x = R/I(x)$ and $M_x = M \otimes_R R_x$ for any R-module M.

Definition ([3] ; p. 92). An R-algebra W is said to be normal if for every $x \in X(R)$ and every connected R_x-algebra Ω (that is, Ω has no idempotents except 0 and 1), all R_x-algebra homomorphisms from W_x to Ω have the same image.

Proof of the Theorem. Since $(S \otimes_R T) \otimes_R R_x \cong S_x \otimes_{R_x} T_x$, $T_x \otimes_{R_x} T_x \cong (T \otimes_R T) \otimes_R R_x \cong T \otimes_R R_x = T_x$ and S_x is a G-Galois extension of R_x, it suffices to show that $S \otimes_R T$ is normal in the case where R is connected. Let Ω be a connected R-algebra and f and g be R-algebra homomorphisms from $S \otimes_R T$ to Ω. Then Ω can be regarded as a T-algebra by defining the multiplication by $\omega * t = \omega f(1 \otimes t)$ for $\omega \in \Omega$ and $t \in T$. Since $\mathrm{Hom}_R(S \otimes_R T, \Omega) = \mathrm{Hom}_T(S \otimes_R T, \Omega)$ by lemma, $\mathrm{Alg}_R(S \otimes_R T, \Omega) = \mathrm{Alg}_T(S \otimes_R T, \Omega)$ and f and g are T-algebra homomorphisms. By Cor. 1. 6 of [1] (p. 88), there exists $\sigma \in G$ such that $f = g(\sigma \otimes 1)$ and $f(S \otimes_R T) = g(\sigma \otimes 1)(S \otimes_R T) = g(S \otimes_R T)$.

Remark. An R-algebra T is said to be purely inseparable if the kernel of the map $T \otimes_R T \to T$ $(t_1 \otimes t_2 \to t_1 t_2)$ is contained in the Jacobson radical of $T \otimes_R T$ ([7] ; p. 343). Hence T in the Theorem is purely inseparable.

In [7], some cases where a subalgebra of the purely inseparable algebra becomes also purely inseparable were studied (Cor. 7 ; p. 347, Prop. 14 ; p. 353). Here we give a simple proposition for this problem.

Proposition. Let S be a purely inseparable R-algebra which is flat and finitely generated as an R-module. Let T be an R-subalgebra of S which is flat as an R-module. Then T is purely inseparable.

Proof. In this case, $T \otimes_R T$ is a subring of $S \otimes_R S$ such that $S \otimes_R S$ is a finitely generated $T \otimes_R T$-module and $J(S \otimes_R S) \cap (T \otimes_R T) = J(T \otimes_R T)$ ([5] ; pp. 90-92), where $J(S \otimes_R S)$ is the Jacobson radical of $S \otimes_R S$. Hence $\ker \pi' = \ker \pi \cap (T \otimes_R T) \subset J(S \otimes_R S) \cap (T \otimes_R T) = J(T \otimes_R T)$, where π' is the map $T \otimes_R T \to T$ $(t_1 \otimes t_2 \to t_1 t_2)$ and π is the map $S \otimes_R S \to S$ $(s_1 \otimes s_2 \to s_1 s_2)$.

References

1) DeMeyer F., and Ingraham, E., 1971. Separable algebras over commutative rings. Lecture notes in math., 181, Springer-Verlag, Berlin.
2) Jacobson, N., 1964. Lectures in abstract algebra III. Theory of fields and Galois theory. Graduate texts in math. 32, Springer-Verlag, New York.
3) Magid, A. R., 1971. Galois groupoid. J. Algebra, **18**, pp., 89-102.
4) Magid, A. R., 1974. The separable Galois theory of commutative rings. Pure and applied math., A series of monographs and textbooks 27, Marcel Dekker, New York.
5) Northcott, D. G., 1968. Lessons on rings, modules and multiplicities. Cambridge

University Press,
6) Silver, L., 1967. Noncommutative localizations and applications. J. Algebra, **7**, pp. 44–76.
7) Sweedler, M. E., 1975. Purely inseparable algebras. J. Algebra, **35**, pp. 342–355.

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体の正規拡大に関連して，可換環の上の正規多元環の例について述べた。