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A Note on Algebras and Semiprime Rings

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Introduction

Throughout this note, we assume that every ring has an identity element 1, every module is unitary, and a subring of a ring contains the same 1. A ring homomorphism will mean one taking 1 to 1. For a ring A, ${}_AM$ (resp. M_A) denotes the left (resp. right) A-module M and $rad\,A$ denotes the prime radical (the intersection of all prime ideals) of A, and A is said to be a semiprime ring if $rad\ A=0$. As well known, A is semiprime if and only if A has no non-zero nilpotent ideals, and when A is commutative, A is semiprime if and only if A has no non-zero nilpotent elements ([5]).

Let R be a commutative ring and A be an R-algebra. A is said to be a separable algebra if ${}_{A\otimes_R A^\circ}A$ is a projective module where A° is the R-algebra opposite to A ([2], p. 40). Especially, when R is a field, A is separable if and only if the dimension of A as a vector space over R is finite and for any field extension L/R, $rad(A\otimes_R L)=0$ ([2], Th. 2. 5, p. 50).

On the other hand, the notion of a separable extension field over a field (not necessarily algebraic) has been defined in [4] (p. 166). That is, when L/K is a field extension and Ω is an algebraic closure of L, L is said to be separable over K if it is either of characteristic 0 or if it is of characteristic $p>0$ and L is lineary disjoint to $K^{p^{-1}}$ over K where $K^{p^{-1}}=\{\omega\in\Omega\,|\,\omega^p\in K\}$. Moreover, we can see that L is separable over K if and only if for any field extension E/K, $rad(L\otimes_K E)=0$ ([4], Th. 21, p. 197 and Th. 23, p. 202).

In [3], [5], [7] and [8], the conditions which insure that a group ring, a twisted group ring, a skew group ring and a crossed product are itself semiprime were studied.

Considering these backgrounds, the purpose of this note is to investigate an algebra A over a commutative ring R which satisfies the following condition (*).

(*) For any R-algebra B such that B is a semiprime ring, $B\otimes_R A$ is also a semiprime ring.

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§ 1. An R-algebra satisfying the condition (*)

Lemma 1. Let A/B be a ring extension. Then we have

$$B\cap rad\,A\subset rad\,B.$$

PROOF. Since $rad\,A$ (resp. $rad\,B$) is the set of all strongly nilpotent elements of A (resp. B) ([5], Prop. 1, p. 56), this is clear.

Lemma 2. (cf. [5], Proof of Lemma 7, p. 164) Let A and B be R-algebras such that the map $B\to A\otimes_R B\ (b\to 1\otimes b)$ is a monomorphism. Then, regarding B as a subring of $A\otimes_R B$, we have

$$B\cap rad(A\otimes_R B)=rad\,B.$$

Hence if $A\otimes_R B$ is semiprime, then so is B.

Proof. By Lemma 1, $B\cap rad(A\otimes_R B)\subset rad\,B$. Let P be a prime ideal of $A\otimes_R B$ and $xBy\subset P\cap B$ for $x,\ y\in B$. Then for any $\sum a_i\otimes b_i\in A\otimes_R B$, we have

$$\begin{aligned} x(\textstyle\sum a_i\otimes b_i)y &= \textstyle\sum a_i\otimes xb_iy \\ &= \textstyle\sum (a_i\otimes 1)(1\otimes xb_iy)\in P. \end{aligned}$$

So $x\in P\cap B$ or $y\in P\cap B$, that is, $P\cap B$ is a prime ideal of B. Hence we have

$$\begin{aligned} rad\,B &\subset \cap(P\cap B) \\ &= B\cap rad(A\otimes_R B). \end{aligned}$$

Proposition 3.

(1) If an R-algebra A satisfies the condition (*), then for any commutative R-algebra S, $S\otimes_R A$ satisfies the condition (*) as an S-algebra.

(2) If a commutative R-algebra S and an S-algebra A satisfy the condition (*), then A satisfies the condition (*) as an R-algebra.

(3) Let S_1 and S_2 be commutative R-algebras. If an S_1-algebra A_1 and an S_2-algebra A_2 satisfy the condition (*), then $A_1\otimes_R A_2$ satisfies the condition (*) as an $S_1\otimes_R S_2$-algebra.

(4) Let S_1 be a commutative R-algebra, and S_2 be a commutative R-algebra which satisfies the condition (*). Let A_1 be an S_1-algebra and A_2 be an S_2-algebra such that $A_1\otimes_R A_2$ satisfies the condition (*) as an $S_1\otimes_R S_2$-algebra. Then, if A_2 is faithful as an R-module and contains R as an R-direct summand, A_1 satisfies the condition (*) as an S_1-algebra. Especially if A_1 and A_2 are R-algebras such that A_2 is faithful as an R-module, A_2 contains R as an R-direct summand and $A_1\otimes_R A_2$ satisfies the condition (*) as an R-algebra, then A_1 satisfies the condition (*).

(5) Let Λ and Γ be R-algebras and $A=\Lambda\oplus\Gamma$ (a direct sum of rings). Then A satisfies the condition (*) as an R-algebra if and only if Λ and Γ

satisfy the condition (*).

(6) Let A_1 be an R_1-algenra and A_2 be an R_2-algebra. If we put $A=A_1\oplus A_2$ and $R=R_1\oplus R_2$ (direct sums of rings), we can regard A as an R-algebra under the multiplication $(r_1, r_2)(a_1, a_2)=(r_1a_1, r_2a_2)$ where $r_1\in R_1$, $r_2\in R_2$, $a_1\in A_1$ and $a_2\in A_2$. Then A satisfies the condition (*) as an R-algebra if and only if R_1-algebra A_1 and R_2-algebra A_2 satisfy the condition (*).

Proof. (1) and (2) are clear.

(3) By (1), $A_1\otimes_R S_2\ (\cong A_1\otimes_{S_1}S_1\otimes_R S_2)$ and $S_1\otimes_R A_2\ (\cong S_1\otimes_R S_2\otimes_{S_2}A_2)$ satisfy the condition (*) as $S_1\otimes_R S_2$-algebras, and hence $A_1\otimes_R A_2\ (\cong A_1\otimes_{S_1}(S_1\otimes_R S_2)\otimes_{S_1\otimes_R S_2}(S_1\otimes_R S_2)\otimes_{S_2}A_2)$ satisfies also the condition (*).

(4) Let B be any S_1-algebra which is a semiprime ring. By the assumptions, $B\otimes_R S_2\ (\cong B\otimes_{S_1}S_1\otimes_R S_2)$ is semiprime, and hence $B\otimes_{S_1}A_1\otimes_R A_2$ $(\cong B\otimes_{S_1}(S_1\otimes_R S_2)\otimes_{S_1\otimes_R S_2}(A_1\otimes_R A_2))$ is also semiprime. Since R is R-direct summand of A_2, the map $B\otimes_{S_1}A_1\to B\otimes_{S_1}A_1\otimes_R A_2\ (b\otimes a_1\to b\otimes a_1\otimes 1)$ is a monomorphism, and by Lemma 2, $B\otimes_{S_1}A_1$ is a semiprime ring.

(5) Let B be any R-algebra. Then since $B\otimes_R A=(B\otimes_R \Lambda)\oplus(B\otimes_R \Gamma)$, we have $rad(B\otimes_R A)=rad(B\otimes_R \Lambda)\oplus rad(B\otimes_R \Gamma)$ and this induces the conclusion.

(6) In this case, R_1 and R_2 satisfy the condition (*) as R-algebras. Let A satisfy the condition (*) as an R-algebra and let B be an R_1-algebra which is a semiprime ring. Then B can be regarded as an R-algebra under the multiplication $(r_1, r_2)b=r_1b$ where $r_1\in R_1$, $r_2\in R_2$ and $b\in B$. Since $B\otimes_R A\cong B\otimes_R A_1\cong B\otimes_{R_1}A_1$, $B\otimes_{R_1}A_1$ is also a semiprime ring. Hence A_1 satisfies the condition (*) as an R_1-algebra and for R_2-algebra A_2, we can take the same manner. The converse follows from (2) and (5).

Proposition 4. Let A be a central separable R-algebra. Then R is semiprime if and only if A is semiprime.

Proof. Let R be a semiprime ring and $\mathfrak{U}$ be any nilpotent ideal of A. If $\mathfrak{U}^m=0$, $(\mathfrak{U}\cap R)^m=0$ and hence $\mathfrak{U}\cap R=0$. Therefore $\mathfrak{U}=(\mathfrak{U}\cap R)A=0$. Conversely, let A be a semiprime ring and $\mathfrak{A}$ be any nilpotent ideal of R. if $\mathfrak{A}^n=0$, $(\mathfrak{A}A)^n=\mathfrak{A}^nA=0$. Hence $\mathfrak{A}A=0$ and $\mathfrak{A}=0$. (cf. [2], Cor. 3. 7, p. 54)

A bimodule ${}_AM_B$ is said to be a Morita module if ${}_AM$ and M_B are finitely generated projective, and $A=\mathrm{End}\ M_B$ and $B=\mathrm{End}\ {}_AM$.

Lemma 5. Let ${}_AM_B$ be a Morita module. Then A is semiprime if and only if B is semiprime.

Proof. It is well known that there exists a one-one correspondence between the set of ideals of A and the set of ideals of B given by the correspondence $\mathfrak{U}\to\mathfrak{B}=\{b\in B \mid Mb\subset\mathfrak{U}M\}$ where $\mathfrak{U}$ is an ideal of A. Hence if B is semiprime and $\mathfrak{U}$ is a nilpotent ideal of A such that $\mathfrak{U}^n=0$, $M\mathfrak{B}^n\subset \mathfrak{U}M\mathfrak{B}^{n-1}\subset\mathfrak{U}^nM=0$. Therefore $\mathfrak{B}^n=0$, so $\mathfrak{B}=0$ and $\mathfrak{U}=\{a\in A \mid aM\subset M\mathfrak{B}=0\}=0$.

Proposition 6. The notion of an R-algebra satisfying the condition (*) is Morita invariant.

Proof. Let A and B be R-algebras and ${}_AM_B$ be a Morita module such that A satisfies the condition (*). Then for any R-algebra Λ, ${}_{A\otimes_R\Lambda}M\otimes_R\Lambda$ is finitely generated projective and

$$\begin{aligned}&\mathrm{Hom}_{B\otimes_R\Lambda}(M\otimes_R\Lambda, M\otimes_R\Lambda)\\ &\cong\mathrm{Hom}_B(M, M)\otimes_R\Lambda\\ &=A\otimes_R\Lambda.\end{aligned}$$

Hence ${}_{A\otimes_R\Lambda}M\otimes_R\Lambda_{B\otimes_R\Lambda}$ is a Morita module. So, if Λ is semiprime, $A\otimes_R\Lambda$ is also semiprime and by Lemma 5, $B\otimes_R\Lambda$ is semiprime.

Proposition 7. Any central separable algebra satisfies the condition (*).

Proof. Let A be a central separable R-algebra. Then A° is also a central separable R-algebra and therefore ${}_{A^\circ\otimes_RA}A^\circ{}_R$ is a Morita module ([2], Th. 3. 4, p. 52). Hence for any R-algebra B, ${}_{A^\circ\otimes_RA\otimes_RB}A^\circ\otimes_RB_B$ is a Morita module, and if B is semiprime, by Lemma 5, $A^\circ\otimes_RA\otimes_RB$ is semiprime. Since R is an R-direct summand of A° ([2], Lemma 3. 1, p. 51), by Lemma 2, $A\otimes_RB$ is a semiprime ring.

We can find something in common between the following proposition and the definition of the separability in [4] (p. 166).

Proposition 8. Let R be a commutative ring of prime characteristic p and S be a commutative R-algebra satisfying the condition (*). Moreover, let Ω/R be a commutative ring extension such that Ω is an S-algebra and a semiprime ring. For any integer $i\geqq 0$, we put $R^{p^{-i}}=\{\omega\in\Omega \mid \omega^{p^i}\in R\}$. Then S is linearly disjoint to $R^{p^{-i}}$ over R.

Proof. Since Ω is commutative and semiprime, $R^{p^{-i}}$ is semiprime. Hence, by the assumption, $S\otimes_RR^{p^{-i}}$ is also semiprime. We consider the map

$$\pi:\ S\otimes_RR^{p^{-i}}\to SR^{p^{-i}}(\textstyle\sum s_j\otimes\omega_j\to\sum s_j\omega_j)\,,$$

and let $\sum s_j\otimes\omega_j\in\mathrm{Ker}\,\pi$ where each $s_j\in S$ and each $\omega_j\in R^{p^{-i}}$. Since p is prime, we have $(\sum s_j\otimes\omega_j)^{p^i}=\sum(s_j\otimes\omega_j)^{p^i}=\sum s_j{}^{p^i}\otimes\omega_j{}^{p^i}=\sum s_j{}^{p^i}\omega_j{}^{p^i}\otimes 1=(\sum s_j\omega_j)^{p^i}\otimes 1=0$. Since $S\otimes_RR^{p^{-i}}$ is semiprime, $\sum s_j\otimes\omega_j=0$ and π is a monomorphism. That is, S is linearly disjoint to $R^{p^{-i}}$ over R.

Corollary 9. Under the same assumptions as in Theorem 8, we put $R^{p^{-\infty}}=\bigcup_{i=0}^{\infty}R^{p^{-i}}$. Then S is linearly disjoint to $R^{p^{-\infty}}$ over R.

Proof. In the proof of Proposition 8, we can replace p^i by p^k where k is the largest integer i such that $\omega_j{}^{p^i}\in R$.

Remark. Let π' be the map $R^{p^{-i}}\otimes_RR^{p^{-i}}\to R^{p^{-i}}$ $(\sum\omega_j\otimes\omega'_j\to\sum\omega_j\omega'_j)$ and π'' be the map $R^{p^{-\infty}}\otimes_RR^{p^{-\infty}}\to R^{p^{-\infty}}$ $(\sum\omega_j\otimes\omega'_j\to\sum\omega_j\omega'_j)$. Then we have $Ker\,\pi'\subset rad(R^{p^{-i}}\otimes_RR^{p^{-i}})$ and $Ker\,\pi''\subset rad(R^{p^{-\infty}}\otimes_RR^{p^{-\infty}})$ by the similar manner to the proof of Proposition 8 and Corollary 9.

§ 2. Examples

Any central separable algebra satisfies the condition (*) (Proposition 7). Moreover let R be a commutative ring and G be a finite group. If the order $|G|$ is a unit in R, then the group ring RG is a separable R-algebra ([2], p. 41) and for any R-algebra B, since $B \otimes_R RG \cong BG$, we can see that RG satisfies the condition (*) as an R-algebra (cf. [5], Prop. 8, p. 162). But the separable algebras do not always satisfy the condition (*). For example, $Z/(4)$ is a separable Z-algebra (cf. [2], Prop. 1. 11, p. 46) but since $Z/(4)$ is not semiprime, $Z/(4)$ does not satisfy the condition (*) as a Z-algebra where Z is the ring of integers. At the same time, this gives an example which insures the fact that the homomorphic image of an algebra satisfying the condition (*) does not always satisfy the condition (*).

In this section, we will give other examples satisfying the condition (*). Lemma 7 of [5] (p. 164) and its proof are our guides.

Let A be a ring, G be a finite group of automorphisms of A and $B = A^G = \{x \in A \mid \sigma(x) = x \ \ for\ all\ \sigma \in G\}$. Then a ring extension A/B is said to be a G-Galois extension if there exist $x_1, \cdots, x_n$; $y_1, \cdots, y_n$ in A such that $\sum_{i=1}^{n} x_i \sigma(y_i) = \delta_{1_\sigma}$, where δ_{1_σ} means Kronecker's delta. In this case, the set $\{(x_i, y_i) \mid i = 1, \cdots, n\}$ is called a G-Galois coordinate system for A/B ([6], p. 116).

Lemma 10. Let A/B be a G-Galois extension such that $A = x_1 B + \cdots + x_n B\,(x_k \in A,\ k = 1, \cdots, n)$. Then there exist $y_1, \cdots, y_n$ in A such that $\{(x_k, y_k) \mid k = 1, \cdots, n\}$ is a G-Galois coordinate system for A/B.

Proof. Let $\{(a_i, a_i') \mid i = 1, \cdots, m\}$ be a G-Galois coordinate system for A/B. We put $a_i = \sum_{k=1}^{n} x_k b_{ki}$ $(i = 1, \cdots, m)$ and $y_k = \sum_{i=1}^{m} b_{ki} a_i'$ $(k = 1, \cdots, n)$. Then for any $\sigma \in G$, we have

$$\begin{aligned}
\sum_{k=1}^{n} x_k \sigma(y_k) &= \sum_{k=1}^{n} x_k \sum_{i=1}^{m} b_{ki} \sigma(a_i') \\
&= \sum_{i=1}^{m} \Big(\sum_{k=1}^{n} x_k b_{ki}\Big) \sigma(a_i') \\
&= \sum_{i=1}^{m} a_i \sigma(a_i') \\
&= \delta_{1_\sigma}.
\end{aligned}$$

Lemma 11. (cf. [9], Th. 9. 10, p. 64) Let A/B be a G-Galois extension such that $G = \{\sigma_1 = 1, \sigma_2, \cdots, \sigma_n\}$. Then, if there exists $a \in A$ such that $\{\sigma_1(a), \cdots, \sigma_n(a)\}$ is a free B-basis of A_B (that is, a is a right G-normal basis element of A/B), then the matrix $(\sigma_i(\sigma_j(a)))$ is regular in the matrix ring $(A)_n$.

Proof. By Lemma 10, there exist $y_1, \cdots, y_n$ in A such that $\{(\sigma_k(a), y_k) \mid k = 1, \cdots, n\}$ is a G-Galois coordinate system for A/B. That is, $\sum_{k=1}^{n} \sigma_k(a)\, \sigma_j(y_k)$

$=\delta_{1j}$. Since $\sum_{k=1}^{n}\sigma_k(a)\sigma_i^{-1}(\sigma_j(y_k))=\delta_{1\sigma_i^{-1}\sigma_j}=\delta_{ij}$, we have $\sum_{k=1}^{n}\sigma_i(\sigma_k(a))\sigma_j(y_k)=\delta_{ij}$, and so

$$\Big(\sigma_i\big(\sigma_j(a)\big)\Big)\Big(\sigma_j(y_i)\Big)=\Big(\sum_{k=1}^{n}\sigma_i\big(\sigma_k(a)\big)\sigma_j(y_k)\Big)$$
$$=(\delta_{ij})\,.$$

On the other hand, we can define a B-homomorphism f_i: $A_B\to B_B\,(x\to \sum_{k=1}^{n}\sigma_k(y_ix))$ and since $\sum_{i=1}^{n}\sigma_i(a)f_i(x)=\sum_{i=1}^{n}\sigma_i(a)\sum_{k=1}^{n}\sigma_k(y_ix)=\sum_{k=1}^{n}(\sum_{i=1}^{n}\sigma_i(a)\sigma_k(y_i))\sigma_k(x)$ $=\sum_{k=1}^{n}\delta_{1k}\sigma_k(x)=\sigma_1(x)=x$, a pair of sets $\{\sigma_1(a),\cdots,\sigma_n(a)\}$ and $\{f_1,\cdots,f_n\}$ is a dual basis of A_B. Therefore

$$\Big(\sigma_j(y_i)\Big)\Big(\sigma_i\big(\sigma_j(a)\big)\Big)=\Big(\sum_{k=1}^{n}\sigma_k(y_i)\,\sigma_k\big(\sigma_j(a)\big)\Big)$$
$$=\Big(\sum_{k=1}^{n}\sigma_k\big(y_i\sigma_j(a)\big)\Big)$$
$$=\Big(f_i\big(\sigma_j(a)\big)\Big)$$
$$=(\delta_{ij})\,.$$

Propositon 12. Let A/R be a G-Galois extension such that R is contained in the center of A. If A has a normal basis over R, then for any R-algebra B, B is semiprime if and only if $A\otimes_R B$ is semiprime.

Proof. Since R is an R-direct summand of A (cf. [2], p. 85), the map $B\to A\otimes_R B\,(b\to 1\otimes b)$ is a monomorphism. Hence by Lemma 2, if $A\otimes_R B$ is semiprime, then B is semiprime. Conversely, let B be semiprime, $G=\{\sigma_1=1,\cdots,\sigma_n\}$ and $\{\sigma_1(a),\cdots,\sigma_n(a)\}$ be a normal basis of A. By Lemma 10, there exist $y_1,\cdots,y_n$ in A such that $\{(\sigma_i(a),\ y_i)\mid i=1,\cdots,n\}$ is a G-Galois coordinate system for A/R. By the automorphism $\sigma_k\otimes 1$: $A\otimes_R B\to A\otimes_R B$ $(x\otimes b\to\sigma_k(x)\otimes b)$, $rad\,(A\otimes_R B)$ is sent onto itself. Hence for any $\sum_{j=1}^{n}\sigma_j(a)\otimes b_j\in$ $rad\,(A\otimes_R B)$, by Lemma 11, we have

$$rad\,(A\otimes_R B)\ni\sum_{k=1}^{n}\sigma_k(y_i)\Big(\sum_{j=1}^{n}\sigma_k\big(\sigma_j(a)\big)\otimes b_j\Big)$$
$$=\sum_{j=1}^{n}\Big(\sum_{k=1}^{n}\sigma_k(y_i)\,\sigma_k\big(\sigma_j(a)\big)\Big)\otimes b_j$$
$$=\sum_{j=1}^{n}\delta_{ij}\otimes b_j$$
$$=b_i\,.$$

Hence by Lemma 2, for any $i\,(i=1,\cdots,n)$, we have $b_i\in B\cap\mathrm{rad}\,(A\otimes_R B)=$ $rad\,B=0$. Therefore $rad\,(A\otimes_R B)=0$.

Proposition 13. (cf. [5], Lemma 7, p. 164) Let S be a commutative separable extension of a semilocal ring R (that is, R contains only finitely many maximal ideals), and we assume that S has no idempotents exept 0 and 1 and S is a finitely generated projective R-module. Then for any R-algebra B, B is semiprime if and only if $S \otimes_R B$ is semiprime.

Proof. Since R is an R-direct summand of S ([2], Cor. 2. 3, p. 94), the map $B \to S \otimes_R B\,(b \to 1 \otimes b)$ is a monomorphism. Hence by Lemma 2, if $S \otimes_R B$ is semiprime, then B is also semiprime. Conversely, let B be semiprime. By Theorem 2. 9 in [2] (p. 98), S can be embedded in a commutative G-Galois extension N of R which has a normal basis ([1], Th. 4. 2, p. 27), where G is the group of all automorphisms of N which leave R elementwise fixed. Since N and S are separable extensions of R which are finitely generated projective R-modules, by Theorem 2. 4 in [2] (p. 94), N is a separable extension of S which is a finitely generated projective S-module. So, S is an S-direct summand of N and the map $S \otimes_R B \to N \otimes_S S \otimes_R B \xrightarrow{\cong} N \otimes_R B$ $(s \otimes b \to 1 \otimes s \otimes b \to s \otimes b)$ is a monomorphism. By Proposition 12, since $N \otimes_R B$ is semiprime, by Lemma 2, $S \otimes_R B$ is also semiprime.

Proposition 14. Let A be an R-algebra satisfying the condition (*) and $A[x]$ be the ring of polynomials in one variable. Then $A[x]$ satisfies the condition (*) as an R-algebra.

Proof. For any R-algebra B, we have $B \otimes_R A[x] \cong (B \otimes_R A)[x]$. Hence when B is semiprime, we have $rad(B \otimes_R A[x]) \cong rad((B \otimes_R A)[x]) = (rad(B \otimes_R A))[x] = 0$ ([5], p. 59).

Corollary 15. Let A be an R-algebra satisfying the condition (*) and $A[x_1, \cdots, x_n]$ be the ring of polynomials in n variables. Then $A[x_1, \cdots, x_n]$ satisfies the condition (*) as an R-algebra, and especially $R[x_1, \cdots, x_n]$ satisfies the condition (*).

References

1) Chase, S. U., D. K. Harrison and A. Rosenberg., 1965. Galois theory and Galois cohomology of commutative rings. Mem. Amer. Math. Soc., 52, pp., 15–33.

2) DeMeyer, F., and E. Ingraham., 1971. Separable algebras over commutative rings. Lecture Notes in Math. 181, Springer-Verlag.

3) Fisher, J. W., and S. Montgomery., 1978. Semiprime skew group rings. J. Alg. 52, pp., 241–247.

4) Jacobson, N., 1961. Lectures in abstract algebra III. Theory of fields and Galois theory. Graduate Texts in Math. 32, Springer-Verlag.

5) Lambek, J., 1976. Lectures on Rings and Modules. Chelsea.

6) Miyashita, Y., 1966. Finite outer Galois theory of non-commutative rings. J. Fac. Sci. Hokkaido Univ. Ser. I. XIX, Nos. 3, 4, pp., 114–134.

7) Passman, D. S., 1978. Crossed product over semiprime rings. Houston J. Math. 4, pp., 583–592.

8) Reid, A., 1976. Semi-prime twisted group rings. J. London Math. Soc. (2), 12, pp., 413–418.
9) Tominaga, H. and T. Nagahara., 1970. Galois theory of simple rings. Okayama Math. Lectures.

要　　約

可換環 R 上の多元環 A で，条件

(*) B は R 上の多元環とし，かつ B は半素環であるとき，$B\otimes_R A$ も半素環である。

を満たすものの性質を調べ，またこのような多元環の例について述べた。