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A Note on Generators and Ring Extensions

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Let A be a ring, A' be a subring of A and ${}_AM$ be a faithful left A-module. We put $\mathrm{End}\,({}_AM)=B$ and $\mathrm{End}\,({}_{A'}M)=B'$. Then B is a subring of B'. In this note, we always assume that $A=\mathrm{End}\,(M_B)$, and for the above case we denote by ${}_{A/A'}M_{B'/B}$.

The purpose of this note is to investigate some properties about ring extension A/A' and B'/B and the module M. In this note, every ring has an identity 1, every ring extension has the common 1 and every module over a ring is unitary.

§ 1. Definitions

(1) ${}_AM|{}_AN$ means that ${}_AM$ is isomorphic to a direct summand of a finite direct sum of copies of ${}_AN$ and ${}_AM\sim{}_AN$ means that ${}_AM|{}_AN$ and ${}_AN|{}_AM$. ${}_AM|{}_AA$ means that ${}_AM$ is finitely generated and projective and ${}_AA|{}_AM$ means that ${}_AM$ is a generator. It is well known that if ${}_AM$ is a generator and $\mathrm{End}\,({}_AM)=B$, then $M_B|B_B$ and $A=\mathrm{End}(M_B)$. ${}_AM$ is said to be a progenerator if ${}_AM\sim{}_AA$, and in this case ${}_AM_B$ is said to be a Morita module.

(2) A ring extension A/A' is said to be a separable extension if the (A, A)-map $\pi_A : {}_AA\otimes_{A'}A_A\to{}_AA_A\,(x\otimes y\to xy)$ splits. It is known that A/A' is a separable extension if and only if ${}_AA_A|{}_AA\otimes_{A'}A_A$. A ring extension A/A' is said to be an H-separable extension if ${}_AA\otimes_{A'}A_A|{}_AA_A$. An H-separable extension is a separable extension ([2], Th. 2.2).

(3) A ring extension A/A' is said to be a Frobenius extension (resp. left QF-extension) if ${}_{A'}A|{}_{A'}A'$ and ${}_AA_{A'}\cong{}_A\mathrm{Hom}_{A'}\,({}_{A'}A, {}_{A'}A')_{A'}$ (resp. ${}_AA_{A'}|{}_A\mathrm{Hom}_{A'}\,({}_{A'}A, {}_{A'}A')_{A'}$). A right QF-extension is similarly defined ([3], [5]).

§ 2.

Proposition 1. In ${}_{A/A'}M_{B'/B}$, we put ${}_B^*M_A={}_B\mathrm{Hom}_A({}_AM, {}_AA)_A$ and ${}_{B'}^{\circledast}M_{A'}={}_{B'}\mathrm{Hom}_{A'}\,({}_{A'}M, {}_{A'}A')_{A'}$. Then the followings are satisfied.

(1) If ${}_{A'}M|{}_{A'}A'$ then ${}_AA\otimes_{A'}M_{B'}\cong{}_A\mathrm{Hom}_B(B'_B, M_B)_{B'}\;(a\otimes m\to(b'\to a\,(mb')))$.

(2) If $M_B|B_B$ then ${}_AM\otimes_BB'_{B'}\cong{}_A\mathrm{Hom}_{A'}\,({}_{A'}A, {}_{A'}M)_{B'}\;(m\otimes b'\to(a\to(am)b'))$.

(3) If $B'_B|B_B$ and ${}_{A'}M|{}_{A'}A'$ then ${}_{B'}B'\otimes_B{}^*M_A\cong{}_{B'}^{\circledast}M\otimes_{A'}A_A$.

Proof. (1) Since ${}_{A'}M|{}_{A'}A'$, by [1] (p. 120),

$$\begin{aligned}{}_AA\otimes_{A'}M_{B'}&={}_A\mathrm{Hom}_B(M_B, M_B)\otimes_{A'}M_{B'}\\&\cong{}_A\mathrm{Hom}_B(\mathrm{Hom}_{A'}\,({}_{A'}M, {}_{A'}M)_B, M_B)_{B'}\\&={}_A\mathrm{Hom}_B(B'_B, M_B)_{B'}.\end{aligned}$$

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(2) This is similar to (1).
(3) By (1) and [1] (p. 120),

$$\begin{aligned}{}_{B'}B'\otimes {}_{B}^{*}M_A &= {}_{B'}B'\otimes_B \mathrm{Hom}_A({}_AM,\ {}_AA)_A\\ &\cong {}_{B'}\mathrm{Hom}_A({}_A\mathrm{Hom}_B(B'_B,\ M_B)_{B'},\ {}_AA)_A\\ &\cong {}_{B'}\mathrm{Hom}_A({}_AA\otimes_{A'}M_{B'},\ {}_AA)_A\\ &\cong {}_{B'}\mathrm{Hom}_{A'}({}_{A'}M,\ {}_{A'}\mathrm{Hom}_A({}_AA,\ {}_AA))_A\\ &\cong {}_{B'}\mathrm{Hom}_{A'}({}_{A'}M,\ {}_{A'}A)_A\\ &\cong {}_{B'}\mathrm{Hom}_{A'}({}_{A'}M,\ {}_{A'}A')\otimes_{A'}A_A\\ &= {}_{B'}^{\circledast}M\otimes_{A'}A_A.\end{aligned}$$

Remark. Let ${}_BM^*{}_A={}_B\mathrm{Hom}_B(M_B,\ B_B)_A$. Then ${}_B^*M_A={}_B\mathrm{Hom}_A({}_AM,\ {}_AA)_A={}_B\mathrm{Hom}_A({}_AM,\ {}_A\mathrm{Hom}_B(M_B,\ M_B))_A\cong {}_B\mathrm{Hom}_B(M_B,\ \mathrm{Hom}_A({}_AM,\ {}_AM)_B)_A={}_B\mathrm{Hom}_B(M_B,\ B_B)_A$ $={}_BM^*{}_A$. And similarly if we put ${}_{B'}M^{\circledast}{}_{A'}={}_{B'}\mathrm{Hom}_{B'}(M_{B'},\ B'_{B'})_{A'}$, then when $A'=\mathrm{End}(M_{B'})$, we have ${}_{B'}^{\circledast}M_{A'}\cong {}_{B'}M^{\circledast}{}_{A'}$ (cf. [4], Lemma 3.1).

Proposition 2. In ${}_{A/A'}M_{B'/B}$, let ${}_AM$ be a generator. Then the followings are equivalent.

(1) A/A' is a separable extension.
(2) The map $\pi_M: {}_AA\otimes_{A'}M_B\to {}_AM_B\ (a\otimes m\to am)$ splits.
(3) The map $\pi_{*M}: {}_B^*M\otimes_{A'}A_A\to {}_B^*M_A\ (g\otimes a\to ga)$ splits.
(4) ${}_AM_B|{}_AA\otimes_{A'}M_B$.
(5) ${}_B^*M_A|{}_B^*M\otimes_{A'}A_A$.

Proof. ((1)$\Longrightarrow$(2)): There exists a map $k_A: {}_AA_A\to {}_AA\otimes_{A'}A_A$ such that $\pi_A\circ k_A=id_A$ (the identity map of A). In the following diagram,

$$\begin{array}{ccc} {}_AA\otimes_{A'}M_B & \xrightarrow{\pi_M} & {}_AM_B\\ \beta\uparrow & & \downarrow\alpha \\ {}_AA\otimes_{A'}A\otimes_AM_B & \underset{k_A\otimes 1}{\overset{\pi_A\otimes 1}{\rightleftarrows}} & {}_AA\otimes_AM_B\end{array}$$

we have $\pi_M=\alpha^{-1}\circ(\pi_A\otimes 1)\circ\beta^{-1}$ where $\alpha(m)=1\otimes m$ and $\beta(a_1\otimes a_2\otimes m)=a_1\otimes a_2m$. Hence if we put $k_M=\beta\circ(k_A\otimes 1)\circ\alpha$, we have $\pi_M\circ k_M=id_M$.

((2)$\Longrightarrow$(1)): There exists a map k_M: ${}_AM_B\to {}_AA\otimes_{A'}M_B$ such that $\pi_M\circ k_M=id_M$. In the following diagram,

$$\begin{array}{ccccc} {}_AA_A & \xrightarrow{\gamma} & {}_A\mathrm{Hom}_B(M_B,\ M_B)_A & \underset{\mathrm{Hom}(1,\ \pi_M)}{\overset{\mathrm{Hom}(1,\ k_M)}{\rightleftarrows}} & {}_A\mathrm{Hom}_B(M_B,\ A\otimes_{A'}M_B)_A\\ & \nwarrow^{\pi_A} & & & \uparrow\phi\\ & & {}_AA\otimes_{A'}A_A & \xleftarrow{\delta} & {}_AA\otimes_{A'}\mathrm{Hom}_B(M_B,\ M_B)_A\end{array}$$

$[\phi(a\otimes g)](m)=a\otimes g(m)$ and γ and δ are canonical maps. Since $M_B|B_B$, ϕ is an isomorphism and since $\pi_A=\gamma^{-1}\circ\mathrm{Hom}(1, \pi_M)\circ\phi\circ\delta^{-1}$, if we put $k_A=\delta\circ\phi^{-1}\circ\mathrm{Hom}(1, k_M)\circ\gamma$, then $\pi_A\circ k_A=id_A$.

$((1)\Leftarrow(3))$: Since ${}^*M_A$ is a generator, $B=\mathrm{End}({}^*M_A)$, ${}_AA_A\cong{}_A\mathrm{Hom}_B({}_B^*M, {}_B^*M)_A\,(a\to(f\to fa))$ and ${}_B^*M\cong{}_BM^*|_BB$, this is proved in the same manner as $((1)\Rightarrow(2))$ and $((2)\Rightarrow(1))$.

$((2)\Rightarrow(4))$ and $((3)\Rightarrow(5))$: This is clear.

$((4)\Rightarrow(1))$ and $((5)\Rightarrow(1))$: In the same manner as $((2)\Rightarrow(1))$, ${}_AA_A={}_A\mathrm{Hom}_B(M_B, M_B)_A|{}_A\mathrm{Hom}_B(M_B, A\otimes_{A'}M_B)_A\cong{}_AA\otimes_{A'}\mathrm{Hom}_B(M_B, M_B)_A={}_AA\otimes_{A'}A_A$ and $((5)\Rightarrow(1))$ is clear.

Remark. (a) If ${}_{A'}M|_{A'}A'$, then by (1) of Propositon 1, ${}_AA\otimes_{A'}M_B\cong{}_A\mathrm{Hom}_B(B'_B, M_B)_B(a\otimes m\to(b'\to a(mb')))$. Hence in this case, in Proposition 2, (2) is equivalent to (2'): The map t_M: ${}_A\mathrm{Hom}_B(B'_B, M_B)_B\to{}_AM_B(f\to f(1))$ splits. (b) In Proposition 2, if ${}_AM$ is a progenerator, the map η: ${}_B^*M\otimes_{A'}A_A\to{}_B\mathrm{Hom}_A({}_AM, {}_AA\otimes_{A'}A)_A$ $(f\otimes a\to(m\to f(m)\otimes a))$ is an isomorphism. So we can define the map $\pi_{*M}\circ\eta^{-1}$: ${}_B\mathrm{Hom}_A({}_AM, {}_AA\otimes_{A'}A)_A\to{}_B^*M_A(h\to\pi_A\circ h)$. Hence A/A' is separable if and only if $\pi_{*M}\circ\eta^{-1}$ splits.

Proposition 3 (cf. [6], Th. 1, (3)). In ${}_{A/A'}M_{B'/B}$, let ${}_AM$ be a progenerator. Then A/A' is a separable extension if and only if ${}_BB_B|{}_B^*M\otimes_{A'}M_B$.

Proof. Let A/A' be separable. Since ${}_AM|_AA$, we have

$$\begin{aligned}{}_BB_B &= {}_B\mathrm{Hom}_A({}_AM, {}_AM)_B\\ &|\,{}_B\mathrm{Hom}_A({}_AM, {}_AA\otimes_{A'}M)_B\\ &\cong {}_B\mathrm{Hom}_A({}_AM, {}_AA)\otimes_{A'}M_B\\ &= {}_B^*M\otimes_{A'}M_B.\end{aligned}$$

Conversely, let ${}_BB_B|{}_B^*M\otimes_{A'}M_B$. Then by above, ${}_BB_B|{}_B\mathrm{Hom}_A({}_AM, {}_AA\otimes_{A'}M)_B$. Hence we have

$$\begin{aligned}{}_AM_B &\cong {}_AM\otimes_BB_B\\ &|\,{}_AM\otimes_B\mathrm{Hom}_A({}_AM, {}_AA\otimes_{A'}M)_B\\ &\cong {}_A\mathrm{Hom}_A({}_A\mathrm{Hom}(M_B, M_B), {}_AA\otimes_{A'}M)_B\\ &= {}_A\mathrm{Hom}_A({}_AA, {}_AA\otimes_{A'}M)_B\\ &\cong {}_AA\otimes_{A'}M_B.\end{aligned}$$

That is, by Proposition 2, A/A' is separable.

Proposition 4. In ${}_{A/A'}M_{B'/B}$, let ${}_AM$ be a generator. Then the followings are equivalent.

(1) A/A' is an H-separable extension.
(2) ${}_AA\otimes_{A'}M_B|{}_AM_B$.
(3) ${}_AA\otimes_{A'}M_B\sim{}_AM_B$.
(4) ${}_B^*M\otimes_{A'}A_A|{}_B^*M_A$.
(5) ${}_B^*M\otimes_{A'}A_A\sim{}_B^*M_A$.

Proof. ((1) =>(2)): ${}_AA\otimes_{A'}M_B \cong {}_AA\otimes_{A'}A\otimes_A M_B | {}_AA\otimes_A M_B \cong {}_AM_B$.
((2) =>(1)): Since $M_B|B_B$, we have

$$\begin{aligned} {}_AA\otimes_{A'}A_A &= {}_AA\otimes_{A'}\mathrm{Hom}_B(M_B, M_B)_A \\ &\cong {}_A\mathrm{Hom}_B(M_B, A\otimes_{A'}M_B)_A \\ &\quad | {}_A\mathrm{Hom}_B(M_B, M_B)_A \\ &= {}_AA_A. \end{aligned}$$

((2)=>(3)): Since an H-separable extension is a separable extension, this is clear by Proposition 2.

Remark. By changing its positions, the above remark and Proposition 3 are valid in H-separable cases.

Corollary 5. (cf. [4], Th. 4.4 and [6], Th. 4] In ${}_{A/A'}M_{B'/B}$, let ${}_AM$ and $M_{B'}$ be generators and $A'=\mathrm{End}(M_{B'})$. Then ${}_{A'}A_{A'}|{}_{A'}A'_{A'}$ (that is, A is a centrally projective A'-module) if and only if B'/B is an H-separable extension.

Proof. By Proposition 4, B'/B is an H-separable extension if and only if ${}_{A'}M\otimes_B B'_{B'}|{}_{A'}M_{B'}$. Let ${}_{A'}A_{A'}|{}_{A'}A'_{A'}$. Since $M_B|B_B$, by (2) of Proposition 1, we have

$$\begin{aligned} {}_{A'}M\otimes_B B'_{B'} &\cong {}_{A'}\mathrm{Hom}_{A'}({}_{A'}A, {}_{A'}M)_{B'} \\ &\quad | {}_{A'}\mathrm{Hom}_{A'}({}_{A'}A', {}_{A'}M)_{B'} \\ &= {}_{A'}M_{B'}. \end{aligned}$$

Hence B'/B is H-separable. Conversely if ${}_{A'}M\otimes_B B'_{B'}|{}_{A'}M_{B'}$, then we have

$$\begin{aligned} {}_{A'}A_{A'} &= {}_{A'}\mathrm{Hom}_B(M_B, M_B)_{A'} \\ &\cong {}_{A'}\mathrm{Hom}_B(M_B, \mathrm{Hom}_{B'}(B'_{B'}, M_{B'})_B)_{A'} \\ &\cong {}_{A'}\mathrm{Hom}_{B'}(M\otimes_B B'_{B'}, M_{B'})_{A'} \\ &\quad | {}_{A'}\mathrm{Hom}_{B'}(M_{B'}, M_{B'})_{A'} \\ &= {}_{A'}A'_{A'}. \end{aligned}$$

Proposition 6. In ${}_{A/A'}M_{B'/B}$, let ${}_AM$ and ${}_{A'}M$ be generators and ${}_{A'}A|{}_{A'}A'$. Then

(1) A/A' is a Frobenius extension if and only if ${}_AM\otimes_B B'_{B'}\cong {}_AA\otimes_{A'}M_{B'}$.
(2) A/A' is a left QF-extension if and only if ${}_AA\otimes_{A'}M_{B'}|{}_AM\otimes_B B'_{B'}$.
(3) A/A' is a right QF-extension if and only if ${}_AM\otimes_B B'_{B'}|{}_AA\otimes_{A'}M_{B'}$ and $A_{A'}|A'_{A'}$.

Proof. (1) Let A/A' be a Frobenius extension. Then by (2) of Proposition 1, we have

$$\begin{aligned} {}_AM\otimes_B B'_{B'} &\cong {}_A\mathrm{Hom}_{A'}({}_{A'}A, {}_{A'}M)_{B'} \\ &\cong {}_A\mathrm{Hom}_{A'}({}_{A'}A, {}_{A'}A')\otimes_{A'}M_{B'} \\ &\cong {}_AA\otimes_{A'}M_{B'}. \end{aligned}$$

Conversely, if ${}_AM\otimes_B B'_{B'}\cong {}_AA\otimes_{A'}M_{B'}$, then we have

$$
\begin{aligned}
{}_{A}\mathrm{Hom}_{A'}({}_{A'}A, {}_{A'}A')_{A'} &= {}_{A}\mathrm{Hom}_{A}({}_{A'}A, {}_{A'}\mathrm{Hom}_{B'}(M_{B'}, M_{B'}))_{A'} \\
&\cong {}_{A}\mathrm{Hom}_{B'}(M_{B'}, \mathrm{Hom}_{A'}({}_{A'}A, {}_{A'}M)_{B'})_{A'} \\
&\cong {}_{A}\mathrm{Hom}_{B'}(M_{B'}, M\otimes_{B}B'_{B'})_{A'} \\
&\cong {}_{A}\mathrm{Hom}_{B'}(M_{B'}, A\otimes_{A'}M_{B'})_{A'} \\
&\cong {}_{A}A\otimes_{A'}\mathrm{Hom}_{B'}(M_{B'}, M_{B'})_{A'} \\
&= {}_{A}A\otimes_{A'}A'_{A'} \\
&\cong {}_{A}A_{A'}.
\end{aligned}
$$

(2) Let A/A' be a left QF-extension. Then in the same manner as (1), we have

$$
\begin{aligned}
{}_{A}A\otimes_{A'}M_{B'} \mid {}_{A}\mathrm{Hom}_{A'}({}_{A'}A, {}_{A'}A')\otimes_{A'}M_{B'} \\
\cong {}_{A}M\otimes_{B}B'_{B'}.
\end{aligned}
$$

And the converse is similar to (1).

(3) By [5] (Satz 2, p. 347), A/A' is a right QF-extension if and only if ${}_{A'}A|_{A'}A'$, $A_{A'}|A'_{A'}$ and ${}_{A}\mathrm{Hom}_{A'}({}_{A'}A, {}_{A'}A')_{A'}|{}_{A}A_{A'}$. Hence this is also similar to (1).

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要　約

A/A' を環境大, ${}_{A}M$ を generator とする。$\mathrm{End}({}_{A}M)=B$, $\mathrm{End}({}_{A'}M)=B'$ とおく。A/A' が分離的, H-分離的, Frobenius, 左 (右) QF の場合に A/A', B'/B, M の性質について述べた。