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A Note on Rational Extensions

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Let A be a ring with identity 1 and M_A be a right unital A-module. In this note, we always assume that A_A is a submodule of M_A, $E(M_A)$ denotes the injective hull of M_A, $T=\mathrm{End}_A(E(M_A))$, $B=\mathrm{End}_T({}_TE(M_A))$, $H=\mathrm{End}_A(E(A_A))$ and $Q=\mathrm{End}_H({}_HE(A_A))$. $E(M_A)$ and $E(A_A)$ become bimodules ${}_TE(M_A)_B$ and ${}_HE(A_A)_Q$ respectively. Q is called the complete ring of right quotients (the right maximal quotient ring) of A ([2], p. 94; [5], p. 4).

For a left A-module ${}_AU$, a subset V of U and a right ideal D of A, we put $\mathrm{Ann}_AV=\{a\in A|av=0$ for all $v\in V\}$ and $\mathrm{Ann}_UD=\{u\in U|du=0$ for all $d\in D\}$.

Let D be a right ideal of A. D is said to be dense if for any $a_1\neq 0$ and $a_2\in A$, there exists $a\in A$ such that $a_1a\neq 0$ and $a_2a\in D$. The Gabriel topology consisting of dense right ideals is denoted by $\mathscr{D}$ ([4], p. 149; [2], p. 96). D is said to be M-dense if for any $m_1\in M$ such that $m_1\neq 0$ and $a_2\in A$, there exists $a\in A$ such that $m_1a\neq 0$ and $a_2a\in D$. Clearly, in our case, A is M-dense and every M-dense right ideal is dense. The Gabriel topology consisting of M-dense right ideals is denoted by $\mathscr{D}_M$.

We prove the following theorem.

Theorem (Therem 17). The following conditions are equivalent.

(1) M_A is a rational extension of A_A.

(2) $B\longrightarrow M_{\mathscr{D}_M}=\varinjlim_{D\in\mathscr{D}_M}\mathrm{Hom}_A(D, M_A)$ $(b\longrightarrow[f])$ is a right A-isomorphism where f is the map $(M: b)\longrightarrow M(x\longrightarrow(1)(bx))$ and $(M: b)=\{x\in A|(1)(bx)\in M\}$, and the direct limit is taken over by the downwards directed family $\mathscr{D}_M$.

(3) $M_{\mathscr{D}_M}$ is a rational extension of B_A.

In this case, $\mathscr{D}=\mathscr{D}_M$. $B=Q$, $M_{\mathscr{D}}$ is a ring and the map of (2) is a ring isomorphism which fixes elements of A.

First, the following facts (1) and (2) are well known.

(1) ([4], p. 149; [2], p. 96) For a right ideal D of A, the following conditions are equivalent.

(a) D is dense.

(b) $\mathrm{Hom}_A(A/D, E(A_A))=0$.

(c) $\mathrm{Ann}_HD=\mathrm{Ann}_HA$.

(d) For any $q\in Q$, $(D: q)$ is a dense right ideal where $(D: q)=\{a\in A|qa\in D\}$.

For any D_1 and $D_2\in\mathscr{D}$ and for any right A-homomorphisms $f_1: D_1\longrightarrow A_A$ and $f_2: D_2\longrightarrow A_A$, we define an equivalence relation $f_1\sim f_2$ when $f_1=f_2$ in $D_1\cap D_2$. $[f]$

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denotes the equivalence class represented by f.

(2) ([2], p. 97) The map $Q \longrightarrow \varinjlim_{D\in\mathscr{D}} \mathrm{Hom}_A(D, A_A)$ $(q \longrightarrow [f])$ is a ring isomorphism which fixes elements of A, where f is the map $(A : q) \longrightarrow A$ $(x \longrightarrow qx)$.

Lemma 1 (cf. [2], p. 94).

(1) The map ϕ: ${}_TT \longrightarrow {}_TE(M_A)$ $(t \longrightarrow t(1))$ is a left T-epimorphism.

(2) The map ψ: $B_A \longrightarrow E(M_A)$ $(b \longrightarrow (1)b)$ is a right A-monomorphism, and $\mathrm{Im}\,\psi$ (the image of ψ) $= \mathrm{Ann}_{E(M_A)}(\mathrm{Ann}_T A)$.

The following Lemma is similar to the above (1) (cf. [3], pp. 228-229).

Lemma 2. For a right ideal D of A, the following conditions are equivalent.

(a) D is M-dense.

(b) $\mathrm{Hom}_A(A/D, E(M_A)) = 0$.

(c) $\mathrm{Ann}_T D = \mathrm{Ann}_T A$.

(d) For any $b \in B$, $(D : b)$ is an M-dense right ideal where $(D : b) = \{a \in A \mid ba \in D\}$.

By Lemma 2, for any $b \in B$, $(M : b) = \{a \in A \mid (1)(ba) \in M\}$ is M-dense since $(A : b)$ is M-dense and $(A : b) \subset (M : b)$.

Lemma 3 (cf. [2], p. 97). Let D be an M-dense right ideal. Then for any $b \in B$, if $bD = 0$ then $b = 0$.

Proof. Let $bD = 0$. By Lemma 1, there exists $t \in T$ such that $(1)b = t(1)$. Hence for any $d \in D$, $0 = (1)(bd) = ((1)b)d = (t(1))d = t(d)$. So $t(D) = 0$ and by Lemma 2, $t(A) = 0$. Hence $0 = t(1) = (1)b$ and by Lemma 1, we have $b = 0$.

To prove the next Lemma 4, we use Lemma 3.

Lemma 4 (cf. [2], p. 97). Let D be an M-dense right ideal. Then we have a left A-isomorphism

$$ {}_A\{b \in B \mid bD \subset A\} \longrightarrow {}_A\mathrm{Hom}_A(D, A_A) \ (b \longrightarrow (d \longrightarrow bd)). $$

Lemma 5 (cf. [4], p. 195)). Let D_1 and D_2 be M-dense right ideals of A and let f be in $\mathrm{Hom}_A(D_1, A_A)$. Then $f^{-1}(D_2) = \{d \in D_1 \mid f(d) \in D_2)$ is an M-dense right ideal of A.

To prove the next Proposition 6, we use Lemma 2, Lemma 4 and Lemma 5. The proof is similar to [2] (p. 97).

Proposition 6 ([4], p. 206). The map $B \longrightarrow \varinjlim_{D\in\mathscr{D}_M} \mathrm{Hom}_A(D, A_A)$ $(b \longrightarrow [f])$ is a ring isomorphism which fixes elements of A, where f is the map $(A : b) \longrightarrow A$ $(x \longrightarrow bx)$ and B can be regarded as a subring of Q.

Let N_A be a submodule of M_A. M_A is said to be a rational extension of N_A if for any $m_1 \neq 0$ and $m_2 \in M$, there exists $a \in A$ such that $m_1 a \neq 0$ and $m_2 a \in N$ ([1], p. 58). For any right ideal D of A, the following statements are trivial.

(1) A_A is a rational extension of D_A if and only if D is dense.

(2) If M_A is a rational extension of D_A, then D is M-dense.

Proposition 7. Let N_A be a submodule of M_A. Then the following conditions are equivalent.

(1) M_A is a rational extension of N_A.

(2) $\mathrm{Hom}_A(M/N, E(M_A))=0$.

(3) $\mathrm{Ann}_T N=\mathrm{Ann}_T M$.

Lemma 8 (cf. [5], p. 2). Let L_A and N_A be submodules of M_A such that $L_A \subset N_A$. Then M_A is a rational extension of L_A if and only M_A is a rational extension of N_A and N_A is a rational xetension of L_A.

Lemma 9. For a right ideal D of A, the following conditions are equivalent.

(1) M_A is a rational extension of D_A.

(2) M_A is a rational extension of A_A and D is M-dense.

(3) M_A is a rational extension of A_A and D is dense.

By Lemma 9, when M_A is a rational extension of A_A, D is dense if and only if D is M-dense. But under the weaker condition that M_A is an essential extension of A_A, D is dense if and only if D is M-dense, since $E(M_A)=E(A_A)$.

Proposition 10. Let D be a right ideal of A and M_A be a rational extension of A_A. Then A_A is a rational extension of D_A if and only if A_A is a rational extension of $(D: b)_A$ for any $b \in B$.

Proof. This follows immediately from Lemma 2 and Lemma 9.

Proposition 11. We have $\mathrm{Hom}_A(B/A, E(M_A))=0$.

Proof. Let f be in $\mathrm{Hom}_A(B_A, E(M_A))$ such that $f|A=0$. Then there exists $t \in T$ such that the following diagram is commutative.

$$\begin{tikzcd} A_A \arrow[r, "k"] \arrow[dr, "f|A"'] & B_A \arrow[r, "\psi"] \arrow[d, "f"] & E(M_A) \arrow[dl, dashed, "t"] \\ & E(M_A) & \end{tikzcd}$$

Here k is the inclusion map. Then for any $b \in B$, $f(b)=t((1)\,b)=(t(1))\,b=(f(1))\,b=0$. Hence we have $f=0$.

Proposition 12. Let ψ be the map defined in Lemma 1 (2). Then $\mathrm{Im}\,\psi \supset M$ if and only if M_A is a rational extension of A_A. In this case, we have $\mathrm{Im}\,\psi=\mathrm{Ann}_{E(M_A)}(\mathrm{Ann}_T M)$ and $\mathrm{Im}\,\psi$ is a maximal rational extension of M_A (as well as A_A).

Proof. By Proposition 6, B_A is a rational extension of A_A. Hence if $\mathrm{Im}\,\psi$ $(\cong B_A) \supset M$, then by Lemma 8, M_A is a rational extension of A_A. In this case, by Proposition 7, we have $\mathrm{Ann}_T M=\mathrm{Ann}_T A$ and hence by Lemma 1, $\mathrm{Im}\,\psi=\mathrm{Ann}_{E(M_A)}$ $(\mathrm{Ann}_T M)$. Conversely, let M_A be a rational extension of A_A. Since $\mathrm{Ann}_T M=\mathrm{Ann}_T A$, we have $\mathrm{Im}\,\psi=\mathrm{Ann}_{E(M_A)}(\mathrm{Ann}_T M) \supset M$. In this case $\mathrm{Im}\,\psi$ is a maximal rational extension of M_A.

Proposition 13. Let N_A be a submodule of M_A such that M_A is a rational extension of N_A. Then we have the following:

(1) For any $m\in M$, $(N: m)=\{a\in A|ma\in N\}$ is an M-dense right ideal.

(2) For any M-dense right ideal D and for any A-homomorphism $f: D_A\longrightarrow M_A$, $f^{-1}(N)=\{d\in D|f(d)\in N\}$ is an M-dense right ideal.

Proof. (1) We prove that $\mathrm{Ann}_T(N: m)\subset\mathrm{Ann}_T A$. Let $t\in\mathrm{Ann}_T(N: m)$. Since $E(M_A)$ is injective, there exists $t'\in T$ such that the following diagram is commutative:

$$\begin{tikzcd} N+mA \arrow[r, "k"] \arrow[d, "\lambda"'] & E(M_A) \arrow[dl, dashed, "t'"] \\ E(M_A) & \end{tikzcd}$$

were $\lambda(n+ma)=t(a)$ and k is the inclusion map. Hence $t'(N)=0$. By Proposition 7, since $\mathrm{Ann}_T N=\mathrm{Ann}_T M$, we have $t'(M)=0$. Hence for any $a\in A$, $t(a)=t(1)\,a=\lambda(m)\,a=t'(m)\,a=0$.

(2) For any $d\in D$, $(f^{-1}(N): d)=\{a\in A|da\in f^{-1}(N)\}=\{a\in A|f(da)\in N\}=\{a\in A|f(d)\,a\in N\}=(N: f(d))$. Hence by (1), $(f^{-1}(N): d)$ is M-dense and $f^{-1}(N)$ is M-dense by the condition of the Gabriel topology.

Corollary 14. Let M_A be a rational extension of A_A and D, D_1 and D_2 be dense right ideals. Then we have the following:

(1) For any $m\in M$, $(D: m)$ is dense.

(2) For any A-homomorphism $f: D_1\longrightarrow M_A$, $f^{-1}(D_2)$ is dense.

We put $A_{\mathscr{D}}=\varinjlim_{D\in\mathscr{D}}\mathrm{Hom}_A(D, A_A)$ $A_{\mathscr{D}_M}=\varinjlim_{D\in\mathscr{D}_M}\mathrm{Hom}_A(D, A_A)$ $M_{\mathscr{D}}=\varinjlim_{D\in\mathscr{D}}\mathrm{Hom}_A(D, M_A)$ and $M_{\mathscr{D}_M}=\varinjlim_{D\in\mathscr{D}_M}\mathrm{Hom}_A(D, M_A)$. $M_{\mathscr{D}}$ (resp. $M_{\mathscr{D}_M}$) is a right $A_{\mathscr{D}}$-(resp. $A_{\mathscr{D}_M}$-) module under the multiplication $[f]\,[\alpha]=[f\circ(\alpha|\alpha^{-1}(D_1))]$ where $f\in\mathrm{Hom}_A(D_1, M_A)$, $\alpha\in\mathrm{Hom}_A(D_2, A_A)$, $D_1\in\mathscr{D}$ and $D_2\in\mathscr{D}$ (resp. $D_1\in\mathscr{D}_M$ and $D_2\in\mathscr{D}_M$) ([4], p. 196). Canonically we have $A\subset A_{\mathscr{D}_M}\subset A_{\mathscr{D}}$ and $M\subset M_{\mathscr{D}_M}$.

$$\begin{tikzcd} & M_{\mathscr{D}} \\ Q\cong A_{\mathscr{D}} \arrow[ur, dash] \arrow[d, dash] & M_{\mathscr{D}_M} \arrow[d, dash] \\ B\cong A_{\mathscr{D}_M} \arrow[ur, dash] \arrow[d, dash] & M \\ A \arrow[ur, dash] & \end{tikzcd}$$

Proposition 15.

(1) $M_{\mathscr{D}_M A}$ is a rational extension of M_A.

(2) For the map $\pi: M\longrightarrow M_{\mathscr{D}}(m\longrightarrow[m])$, $M_{\mathscr{D}}$ is a rational extension of $\pi(M)_A$.

Proof. (1) Let $[f_1]$ and $[f_2]$ be in $M_{\mathscr{D}_M}$ such that $[f_1]\neq 0$ where $f_1\in\mathrm{Hom}_A(D_1, \mathrm{M}_A)$ and $f_2\in\mathrm{Hom}_A(D_2, M_A)$. Then there exists $d\in D_1\cap D_2$ such that $f_1(d)\neq 0$, and we have $[f_1]\,d\neq 0$ and $[f_2]\,d\in M$.

(2) This is simiiar to (1).

Lemma 16 ([2], p. 97). Let M_A be a rational extension of A_A and D be a dense right ideal. then we have an isomorphism

$$\{b\in B|(1)\,bd\in M \text{ for all } d\in D\}\longrightarrow \mathrm{Hom}_A(D, M_A)\ (b\longrightarrow(d\longrightarrow(1)\,bd)).$$

Theorem 17. The following conditions are equivalent.

(1) M_A is a rational extension of A_A.

(2) $B\longrightarrow M_{\mathscr{D}_M}(b\longrightarrow[f])$ is a right A-isomorphism where f is the map $(M\colon b)\longrightarrow M(x\longrightarrow(1)\,(bx))$.

(3) $M_{\mathscr{D}_M}$ is a rational extension of B_A.

In this case, $\mathscr{D}=\mathscr{D}_M$, $B=Q$, $M_{\mathscr{D}}$ is a ring and the map of (2) is a ring isomorphism which fixes elements of A.

Proof. (1)$\Rightarrow$(2): Let M_A be a rational extension of A_A. Then by Lemma 9, we have $\mathscr{D}=\mathscr{D}_M$. Hence Q equals B and by Proposition 15, $M_{\mathscr{D}A}$ is the maximal rational extension of A_A. So we have $B_A\cong M_{\mathscr{D}A}(b\longrightarrow[f])$ where f is the map $(M\colon b)\longrightarrow M(x\longrightarrow(1)\,(bx))$ (cf. Lemma 16). (2)$\Rightarrow$(1): Since B_A is a rational extension of A_A, $M_{\mathscr{D}_M}$ is a rational extension of A_A and hence M_A is a rational extension of A_A. (1)$\Leftrightarrow$(3): This is trivial by Lemma 8 and Proposition 15. In this case, for any $f_1: D_1\longrightarrow M$ and $f_2: D_2\longrightarrow M(D_1, D_2\in\mathscr{D})$, we define $[f_1]\,[f_2]=[f_1\circ(f_2|f_2^{-1}(D_1))]$ and $[f_1]+[f_2]=[f_1+f_2]$ where $f_1+f_2: D_1\cap D_2\longrightarrow M(d\longrightarrow f_1(d)+f_2(d))$. Then $M_{\mathscr{D}}$ becomes a ring and the map of (2) is a ring isomorphism which fixes elements of A.

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要　　約

A を単位元をもつ環，M_A を A_A を部分加群として含む右 A-加群，$\mathscr{D}_M$ を A の M-dense 右 ideal 全体のなす Gabriel topology とする。M_A が A_A 上 rational であること，$E(M_A)$ (M_A の injective hull) の double centralizer, $M_{\mathscr{D}_M}=\varinjlim_{D\in\mathscr{D}_M}\mathrm{Hom}_A(D, M_A)$ の関係について調べた。