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Note on Ideal Algebras and Gabriel Topologies

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In this note, we assume that all rings have an identity 1 and all modules over a ring are unitary.

Let R be a commutative ring and A be an R-algebra. A is called an ideal algebra over R if for any ideal I' of A and for any ideal I of R, the conditions (*) $(I'\cap R)A=I'$ and $IA\cap R=I$ are satisfied where $I'\cap R$ is the inverse image of I' with respect to the ring homomorphism $R\to A$ ([5]). An ideal algebra is a faithfully flat R-module, and a central separable algebra is an example of an ideal algebra ([5]).

A non-empty family of right ideals $\mathfrak{f}'$ of A is called a Gabriel topology when the following conditions are satisfied.

(1) If $I^*\in\mathfrak{f}'$ and $a\in A$, then $(I^*:a)\in\mathfrak{f}'$, where $(I^*:a)=\{x\in A\mid ax\in I^*\}$.

(2) Let I^* be a right ideal of A. If there exists $J^*\in\mathfrak{f}'$ such that for every $a\in J^*$, $(I^*:a)\in\mathfrak{f}'$ then $I^*\in\mathfrak{f}'$.

A non-empty family of ideals $\mathfrak{L}'$ of A is called a strong filter when the following conditions are satisfied.

(1) If $I'\in\mathfrak{L}'$ and J' is any ideal such that $I'\subset J'$, then $J'\in\mathfrak{L}'$.

(2) If $I'\in\mathfrak{L}'$ and $J'\in\mathfrak{L}'$, then $I'J'\in\mathfrak{L}'$.

By (1) and (2), it is satisfied that if I' and $J'\in\mathfrak{L}'$ then $I'\cap J'\in\mathfrak{L}'$.

§1. Strong filters

Let $\mathfrak{L}'$ be a strong filter of A. For a right A-module M_A, we let $M_{(\mathfrak{L}')}=\varinjlim_{I'\in\mathfrak{L}'}\mathrm{Hom}_A(I'_A, M_A)$. $A_{(\mathfrak{L}')}$ is a ring and $M_{(\mathfrak{L}')}$ is a right $A_{(\mathfrak{L}')}$-module (cf. [1]).

Lemma 1.

(1) Let M_A be a finitely generated and projective A-module, N_B be a right B-module and ${}_AU_B$ be a bimodule. Then the following map

$\Phi : M\otimes_A\mathrm{Hom}_B(N_B, U_B)\to\mathrm{Hom}_B(N_B, M\otimes_A U_B)$ $(m\otimes f\to(x\to m\otimes f(x)))$

is an isomorphism where $m\in M$, $f\in\mathrm{Hom}_B(N_B, U_B)$ and $x\in N$. The inverse map Φ^{-1} is defined as $\Phi^{-1}(h)=\sum_{i=1}^{n} m_i\otimes(\hat{g}_i\circ h)$ where $\{m_i, g_i\}(1\leqq i\leqq n)$ is a dual basis for M_A, $h\in\mathrm{Hom}_B(N_B, M\otimes_A U_B)$ and $\hat{g}_i$ is the map $M\otimes_A U_B\to U_B$ $(m\otimes u\to g_i(m)u)$ $(u\in U)$.

(2) Let A be an R-algebra and M_A be a right A-module which is finitely generated and projective as an R-module. Then for any X_R and N_R, the following map

Ψ: $\mathrm{Hom}_R(N_R, X_R)\otimes_R M_A\to\mathrm{Hom}_A(N\otimes_R A_A, X\otimes_R M_A)_A$ $(f\otimes m\to(n\otimes a\to f(n)\otimes ma))$ is an isomorphism as right A-modules where $f\in\mathrm{Hom}_R(N_R,X_R)$, $m\in M$, $n\in N$ and $a\in A$. The

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inverse map Ψ^{-1} is defined as $\Psi^{-1}(h)=\sum_{i=1}^{s}(\hat{k}_i\circ(h|N))\otimes m_i$ where $\{m_i, k_i\}(1\leqq i\leqq s)$ is a dual basis for M_R, $(h|N)$ is the map $N\to X\otimes_R M$ $(n\to h(n\otimes 1))$ and $\hat{k}_i$ is the map $X\otimes_R M\to X$ $(x\otimes m\to xk_i(m))$ $(x\in X,\ h\in \mathrm{Hom}_A(N\otimes_R A_A, X\otimes_R M_A))$.

Proposition 2.

Let A be an ideal algebra over R. Then there is a one-to-one correspondence between strong filters $\mathfrak{L}'$ of A and strong filters $\mathfrak{L}$ of R.

Proof.

By the relation (*), for a strong filter $\mathfrak{L}'$ of A, $\mathfrak{L}=\{I'\cap R\,|\,I'\in\mathfrak{L}'\}$ is a strong filter of R, and for a strong filter $\mathfrak{L}$ of R, $\mathfrak{L}'=\{IA\,|\,I\in\mathfrak{L}\}$ is a strong filter of A.

Proposition 3.

Let A be an ideal algebra over R and let $\mathfrak{L}'$ and $\mathfrak{L}$ be corresponding strong filters in Proposition 2.

(1) If M_A is finitely generated and projective as an R-module, then there is an isomorphism $M_{(\mathfrak{L}')}\cong R_{(\mathfrak{L})}\otimes_R M$, and if A_R is finitely generated and projective, this isomorphism is a right $A_{(\mathfrak{L}')}$-isomorphism.

(2) If M_A is finitely generated and projective, then there is an isomorphism $M_{(\mathfrak{L}')}\cong M\otimes_A A_{(\mathfrak{L}')}$ as right $A_{(\mathfrak{L}')}$-modules.

proof.

(1) Since A_R is flat and M_R is finitely generated and projective, by Lemma 1 (2), we have

$$\begin{aligned} M_{(\mathfrak{L}')} &= \varinjlim_{I'\in\mathfrak{L}'} \mathrm{Hom}_A(I'_A, M_A)\\ &= \varinjlim_{I\in\mathfrak{L}} \mathrm{Hom}_A(IA_A, M_A)\\ &\cong \varinjlim_{I\in\mathfrak{L}} \mathrm{Hom}_A(I\otimes_R A_A, M_A)\\ &\cong \varinjlim_{I\in\mathfrak{L}} \mathrm{Hom}_R(I_R, R_R)\otimes_R M\\ &= R_{(\mathfrak{L})}\otimes_R M \end{aligned}$$

(2) By Lemma 1 (1), we have

$$\begin{aligned} M_{(\mathfrak{L}')} &= \varinjlim_{I'\in\mathfrak{L}'} \mathrm{Hom}_A(I'_A, M_A)\\ &\cong \varinjlim_{I'\in\mathfrak{L}'} M\otimes_A \mathrm{Hom}_A(I'_A, A_A)\\ &\cong M\otimes_A \varinjlim_{I'\in\mathfrak{L}'} \mathrm{Hom}_A(I'_A, A_A)\\ &= M\otimes_A A_{(\mathfrak{L}')}, \end{aligned}$$

§ 2. Gabriel topologies

Lemma 4.

Let A be an ideal algebra over R and $\mathfrak{f}'$ be a Gabriel topology of A. We set $\mathfrak{L}'=\{I'\in\mathfrak{f}'\,|\,I' \text{ is an ideal}\}$. Then $\mathfrak{L}'$ is a strong filter of A and $\mathfrak{L}=\{I'\cap R\,|\,I'\in\mathfrak{L}'\}$ is a Gabriel topology of R. Further we obtain $\mathfrak{L}=\{AI^*\cap R\,|\,I^*\in\mathfrak{f}'\}=\{I\,|\,I \text{ is an ideal of R and } IA\in\mathfrak{L}'\}$.

Lemma 5.

Let A be an ideal algebra over R and $\mathfrak{L}$ be a Gabriel topology of R. We set $\mathfrak{L}'=\{IA \mid I\in\mathfrak{L}\}$. Let $\mathfrak{f}'$ be the class of right ideals of A which is the weakest Gabriel topology of A containing $\mathfrak{L}'$. Then.

(1) $\mathfrak{f}'=\{I^* \mid I^*$ is a right ideal of A and there exists $I\in\mathfrak{L}$ such that $IA\subset I^*\}$,

(2) $\mathfrak{L}'=\{I'\in\mathfrak{f}' \mid I'$ is an ideal$\}$ and $\mathfrak{L}'$ is a strong filter,

(3) $\mathfrak{L}=\{I'\cap R \mid I'\in\mathfrak{L}'\}=\{I^*\cap R \mid I^*\in\mathfrak{f}'\}$,

(4) $\mathfrak{f}'=\{I^* \mid I^*$ is a right ideal of A and $I^*\cap R\in\mathfrak{L}\}$.

Proof.

(1) We set $\mathfrak{L}^*=\{I^* \mid I^*$ is a right ideal of A and there exists $I\in\mathfrak{L}$ such that $IA\subset I^*\}$. Clearly $\mathfrak{L}^*\subset\mathfrak{f}'$. We prove that $\mathfrak{L}^*$ is a Gabriel topology.

Let $I^*\in\mathfrak{L}^*$, $IA\subset I^*$ and $a\in A$. Then since $IA\subset(I^*:a)$, $(I^*:a)\in\mathfrak{L}^*$.

Next, let $\hat{I}$ be a right ideal of A and J^* be in $\mathfrak{L}^*$ such that for any $a^*\in J^*$, $(\hat{I}:a^*)\in\mathfrak{L}^*$. There exists $J\in\mathfrak{L}$ such that $JA\subset J^*$. For any $j\in J$, $(\hat{I}\cap R:j)=\{r\in R \mid jr\cdot 1\in\hat{I}\}=(\hat{I}:j\cdot 1)\cap R$, and $(\hat{I}:j\cdot 1)\in\mathfrak{L}^*$, Hence there exists $I\in\mathfrak{L}$ such that $IA\subset(\hat{I}:j\cdot 1)$. Since $I=IA\cap R\subset(\hat{I}:j\cdot 1)$, we have $(\hat{I}:j\cdot 1)\cap R\in\mathfrak{L}$ and so $\hat{I}\cap R\in\mathfrak{L}$. Hence $(\hat{I}\cap R)A\subset\hat{I}\in\mathfrak{L}^*$. Therefore $\mathfrak{L}^*$ is a Gabriel topology. Since $\mathfrak{f}'$ is weakest, we obtain $\mathfrak{f}'=\mathfrak{L}^*$.

(2), (3) and (4) are easily seen by (1).

Let $\mathfrak{f}'$ be a Gabriel topology of A and $\triangle$ be a subset of $\mathfrak{f}'$. $\triangle$ is said to be a basis of $\mathfrak{f}'$ if for any $I^*\in\mathfrak{f}'$, there exists $\tilde{I}\in\triangle$ such that $\tilde{I}\subset I^*$ ([7], p. 145). A Gabriel topology is called bounded if it has a basis consisting of ideals ([7], p. 150).

Proposition 6.

Let A be an ideal algebra over R. Then there is a one-to-one correspondence between bounded Gabriel topologies of A and Gabriel topologies of R.

Proof.

Let $\mathfrak{f}'$ be a bounded Gabriel topology of A and $\mathfrak{L}'$ be the set of ideals in $\mathfrak{f}'$. Then $\mathfrak{L}'$ is a basis of $\mathfrak{f}'$ and by Lemma 4, $\mathfrak{L}=\{I'\cap R \mid I'\in\mathfrak{L}'\}$ is a Gabriel topology of R. Further, by Lemma 5, $\mathfrak{L}'=\{IA \mid I\in\mathfrak{L}\}$ and we can see that $\mathfrak{f}'=\{I^* \mid I^*$ is a right ideal of A and there exists $I\in\mathfrak{L}$ such that $IA\subset I^*\}$.

On the other hand, let $\mathfrak{L}$ be a Gabriel topology of R. Then by Lemma 5, $\{I^* \mid I^*$ is a right ideal of A and there exists $I\in\mathfrak{L}$ such that $IA\subset I^*\}$ is a bounded Gabriel topology, and we can see that $\mathfrak{L}=\{I'\cap R \mid I'\in\mathfrak{f}'$ and I' is an ideal$\}$.

Example.

Let A be an ideal algebra over R. Then it is easily seen that if P is a prime ideal of R then PA is a prime ideal of A and if P' is a prime ideal of A then $P'\cap R$ is a prime ideal of R. For a prime ideal P of R, $\mathfrak{L}_P=\{I \mid I$ is an ideal of R such that $I\not\subset P\}$ is a Gabriel topology ([7] p. 151). Then the corresponding strong filter and bounded Gabriel topology of A are $\mathfrak{L}'_{PA}=\{I' \mid I'$ is an ideal of A such that $I'\not\subset PA\}$ and $\mathfrak{f}'_{PA}=\{I^* \mid I^*$ is a right ideal and there exists an ideal $I'\subset I^*$ such that $I'\not\subset PA\}$. More generally, for a set of prime ideals $\mathfrak{p}$ of R, $\bigcap_{P\in\mathfrak{p}}\mathfrak{L}_P$ is a Gabriel topology and the corresponding strong filter and bounded Gabriel topology are $\bigcap_{P\in\mathfrak{p}}\mathfrak{L}'_{PA}$ and $\bigcap_{P\in\mathfrak{p}}\mathfrak{f}'_{PA}$.

For a right A-module M_A and a Gabriel topology $\mathfrak{f}'$ of A, we define

$M_{(\mathfrak{f}')} = \varinjlim_{I^* \in \mathfrak{f}'} \mathrm{Hom}_A(I^*_A, M_A)$ and $M_{\mathfrak{f}'} = \varinjlim_{I^* \in \mathfrak{f}'} \mathrm{Hom}_A(I^*_A, M/t'(M)_A)$ where $t'(M)$ is the torsion submodule of M with respect to $\mathfrak{f}'$ ([7], pp. 195-197).

Proposition 7.

Let A be an ideal algebra over R. $\mathfrak{f}'$ be a bounded Gabriel topology of A and $\mathfrak{L}$ be the Gabriel topology of R corresponding to $\mathfrak{f}'$. Then,

(1) (cf. [4], p. 521, Cor. 2.9) if M_A is finitely generated and projective as an R-module, then there are isomorphisms $M_{(\mathfrak{f}')} \cong M \otimes_R R_{(\mathfrak{L})}$ and $M_{\mathfrak{f}'} \cong M \otimes_R R_{\mathfrak{L}}$. If A is finitely generated and projective as an R-module, these isomorphisms are isomorphisms as right $A_{(\mathfrak{f}')}$-, $A_{\mathfrak{f}'}$-modules respectively.

(2) (cf. [2], p. 32, Th. 4.7) if M_A is finitely generated and projective, then there are isomorphisms $M_{(\mathfrak{f}')} \cong M \otimes_A A_{(\mathfrak{f}')}$ and $M_{\mathfrak{f}'} \cong M \otimes_A A_{\mathfrak{f}'}$ as right $A_{(\mathfrak{f}')}$-, $A_{\mathfrak{f}'}$-modules respectively.

Proof.

(1) By the definition and Proposition 3 (1), $M_{(\mathfrak{f}')} = \varinjlim_{I^* \in \mathfrak{f}'} \mathrm{Hom}_A(I^*_A, M_A)$, and $M_{(\mathfrak{L}')} \cong R_{(\mathfrak{L})} \otimes_R M$. Further, the map $M_{(\mathfrak{L}')} = \varinjlim_{IA \in \mathfrak{L}'} \mathrm{Hom}_A(IA_A, M_A) \to M_{(\mathfrak{f}')}$ $([h] \to [h])$ is an isomorphism. Since M_R is finitely generated and projective, $t'(M) = M \otimes_R t(R) = t(R)M$ where $t(R)$ is the torsion submodule of R with respect to $\mathfrak{L}$. Hence by Lemma 1 (2), we have

$$\begin{aligned} M_{\mathfrak{f}'} &= \varinjlim_{I^* \in \mathfrak{f}'} \mathrm{Hom}_A(I^*_A, M/t'(M)_A) \\ &\cong \varinjlim_{I \in \mathfrak{L}} \mathrm{Hom}_A(IA_A, R/t(R) \otimes_R M_A) \\ &\cong M \otimes_R \varinjlim_{I \in \mathfrak{L}} \mathrm{Hom}_R(I, R/t(R)) \\ &\cong M \otimes_R R_{\mathfrak{L}}. \end{aligned}$$

(2) The fact that $M_{(\mathfrak{f}')} \cong M \otimes_A A_{(\mathfrak{f}')}$ is proved in a way similar to that of Proposition 3 (2). Since M_A is finitely generated and projective, $t'(M) = M \otimes_A t'(A) = Mt'(A)$ and hence by Lemma 1 (1), we have

$$\begin{aligned} M_{\mathfrak{f}'} &= \varinjlim_{I^* \in \mathfrak{f}'} \mathrm{Hom}_A(I^*_A, M/t'(M)_A) \\ &\cong \varinjlim_{I^* \in \mathfrak{f}'} \mathrm{Hom}_A(I^*_A, M \otimes_A A/t'(A)_A) \\ &\cong M \otimes_A \varinjlim_{I^* \in \mathfrak{f}'} \mathrm{Hom}_A(I^*_A, A/t'(A)_A) \\ &\cong M \otimes_A A_{\mathfrak{f}'}. \end{aligned}$$

§ 3. Torsion theories

Let A be an ideal algebra over R, $\mathfrak{f}'$ be a bounded Gabriel topology of A and $\mathfrak{L}$ be the Gabriel topology of R corresponding to $\mathfrak{f}'$. Let $(\mathfrak{T}', \mathfrak{F}')$ be the torsion theory of A with respect to $\mathfrak{f}'$ and $(\mathfrak{T}, \mathfrak{F})$ be the torsion theory of R with respect to $\mathfrak{L}$.

Proposition 8. (cf. [4] , p. 515, Th 2.5)

(1) For a right A-module M_A, $M_A \in \mathfrak{T}'$ if and only if $M_R \in \mathfrak{T}$. Hence $\mathfrak{T}' = \{M_A \mid M_R \in \mathfrak{T}\}$.

(2) For a right A-module M_A, if $M_R \in \mathfrak{T}$ then for any left A-module ${}_AX$, $M \otimes_A X_R \in \mathfrak{T}$.

(3) For an R-module N_R, if $N_R \in \mathfrak{T}$ then for any right A-module X_A, $N \otimes_R X_A \in \mathfrak{T}'$.

Conversely, if $N\otimes_R A_A \in \mathfrak{T}'$ then $N_R \in \mathfrak{T}$. Hence we have $\mathfrak{T} = \{N_R \mid N\otimes_R A_A \in \mathfrak{T}'\} = \{N_R \mid \text{for any } X_A,\ N\otimes_R X_A \in \mathfrak{T}'\}$.

(4) For an R-module N_R, if $N_R \in \mathfrak{T}$ then for any R-module X, $N\otimes_R X_R \in \mathfrak{T}$.

Proof.

(1) Let $M_A \in \mathfrak{T}'$. For any $m \in M$, $\mathrm{Ann}_A(m) = \{a \in A \mid ma = 0\} \in \mathfrak{f}'$. By Lemma 5 and Proposition 6, there exists $I \in \mathfrak{L}$ such that $IA \subset \mathrm{Ann}_A(m)$. Hence $I \subset \mathrm{Ann}_R(m)$ and $M_R \in \mathfrak{T}$. Conversely, let $M_R \in \mathfrak{T}$. For any $m \in M$, $\mathrm{Ann}_R(m) \in \mathfrak{L}$. Since $(\mathrm{Ann}_R(m))A \subset \mathrm{Ann}_A(m)$, by Lemma 5, we have $\mathrm{Ann}_A(m) \in \mathfrak{f}'$ and $M_A \in \mathfrak{T}'$.

(2) For any R-module $F_R \in \mathfrak{F}$, since $\mathrm{Hom}_R({}_R M\otimes_A X, {}_R F) \cong \mathrm{Hom}_A({}_A X, \mathrm{Hom}_R({}_R M, {}_R F)) = 0$, we have $M\otimes_A X \in \mathfrak{T}$.

(3) Let $N_R \in \mathfrak{T}$. For any X_A, we will prove $N\otimes_R X_R \in \mathfrak{T}$. For any $F_R \in \mathfrak{F}$, since $\mathrm{Hom}_R(N\otimes_R X_R,\ F_R) \cong \mathrm{Hom}_R(X\otimes_R N_R,\ F_R) \cong \mathrm{Hom}_R(X_R,\ \mathrm{Hom}_R(N_R,\ F_R)) = 0$, we have $N\otimes_R X_R \in \mathfrak{T}$. Hence by (1), $N\otimes_R X_A \in \mathfrak{T}'$. Conversely, by the assumption, $N\otimes_R A_A \in \mathfrak{T}'$ and by (1), $N\otimes_R A_R \in \mathfrak{T}$. Since A_R is faithfully flat, the map $N \to N\otimes_R A$ $(n \to n\otimes 1)$ is an R-monomorphism. Hence for any $n \in N$, $\mathrm{Ann}_R(n) = \mathrm{Ann}_R(n\otimes 1) \in \mathfrak{L}$ and $N_R \in \mathfrak{T}$.

(4) This is similar to (3).

Proposition 9. (cf. [3])

(1) For a right A-module M_A, $M_A \in \mathfrak{F}'$ if and only if $M_R \in \mathfrak{F}$. Hence $\mathfrak{F}' = \{M_A \mid M_R \in \mathfrak{F}\}$.

(2) For an R-module N_R, if $N\otimes_R A_A \in \mathfrak{F}'$ then $N_R \in \mathfrak{F}$.

(3) For an R-module N_R, if $N_R \in \mathfrak{F}$ then for any right A-module X_A which is finitely generated and projective as an R-module, $N\otimes_R X_A \in \mathfrak{F}'$. Hence if A_R is finitely generated and projective then $\mathfrak{F} = \{N_R \mid N\otimes_R A_A \in \mathfrak{F}'\}$.

Proof.

(1) Let $M_A \in \mathfrak{F}'$. For any $N_R \in \mathfrak{T}$, by Proposition 8 (3), $N\otimes_R A_A \in \mathfrak{T}'$ and since $0 = \mathrm{Hom}_A(N\otimes_R A_A,\ M_A) \cong \mathrm{Hom}_R(N_R,\ \mathrm{Hom}_A(A_A, M_A)) \cong \mathrm{Hom}_R(N_R,\ M_R)$, we have $M_R \in \mathfrak{F}$. Conversely, let $M_R \in \mathfrak{F}$. For any $N_A \in \mathfrak{T}'$, by Proposition 8 (1), $N_R \in \mathfrak{T}$. Hence $\mathrm{Hom}_A(N_A,\ M_A) \subset \mathrm{Hom}_R(N_R, M_R) = 0$, and so $M_A \in \mathfrak{F}'$.

(2) By (1), $N\otimes_R A_R \in \mathfrak{F}$. Since A_R is faithfully flat, the map $N \to N\otimes_R A$ $(n \to n\otimes 1)$ is an R−monomorphism. Since $\mathfrak{F}$ is closed under submodules, we have $N_R \in \mathfrak{F}$.

(3) Let $N_R \in \mathfrak{F}$. Since X_R is finitely generated and projective, by Lemma 1 (1), for any $T_R \in \mathfrak{T}$, $\mathrm{Hom}_R(T_R,\ N\otimes_R X_R) \cong \mathrm{Hom}_R(T_R,\ N_R)\otimes_R X = 0$. Hence $N\otimes_R X_R \in \mathfrak{F}$, so by (1), $N\otimes_R X_A \in \mathfrak{F}'$.

Proposition 10.

(1) (cf. [4], Th. 2.5) Let N_R be an R-module. If $N_R \in \mathfrak{F}$ then for any left A-module ${}_A X$, $\mathrm{Hom}_R(X_R,\ N_R)_A \in \mathfrak{F}'$.

(2) Let F_A be a right A-module. If $F_R \in \mathfrak{F}$ then for any right A-module X_A, $\mathrm{Hom}_A(X_A,\ F_A)_R \in \mathfrak{F}$.

(3) Let A_R be finitely generated and projective. If $\mathrm{Hom}_R(A_R,\ N_R)_A \in \mathfrak{F}'$ then $N_R \in \mathfrak{F}$. Hence if A_R is finitely generated and projective, $\mathfrak{F} = \{N_R \mid \mathrm{Hom}_R(A_R,\ N_R)_A \in \mathfrak{F}'\}$.

Proof.

(1) For any right A-module $M_A \in \mathfrak{T}'$, by Proposition 8 (1) and (2), $M_R \in \mathfrak{T}$ and $M\otimes_A X_R \in \mathfrak{T}$. Hence $\mathrm{Hom}_A(M_A,\ \mathrm{Hom}_R(X_R,\ N_R)_A) \cong \mathrm{Hom}_R(M\otimes_A X_R,\ N_R) = 0$, so we have $\mathrm{Hom}_R(X_R,\ N_R)_A \in \mathfrak{F}'$.

(2) For any $N_R \in \mathfrak{T}$, $\mathrm{Hom}_R(N_R, \mathrm{Hom}_A(X_A, F_A)) \cong \mathrm{Hom}_A(X_A, \mathrm{Hom}_R(N_R, F_R)) = 0$. Hence we have $\mathrm{Hom}_A(X_A, F_A) \in \mathfrak{F}$.

(3) By Proposition 9 (1), $\mathrm{Hom}_R(A_R, N_R)_R \in \mathfrak{F}$ and by Lemma 1 (1) and Proposition 9 (3), $\mathrm{Hom}_R(A_R, N \otimes_R A_R)_A \cong \mathrm{Hom}_R(A_R, N_R) \otimes_R A_A \in \mathfrak{F}'$. Hence by Proposition 9 (1), $\mathrm{Hom}_R(A_R, N \otimes_R A_R)_R \in \mathfrak{F}$. For any R-module $T_R \in \mathfrak{T}$ and $g \in \mathrm{Hom}_R(T_R, N_R)$, we have the following commutative diagram

$$\begin{tikzcd}
T_R \arrow[r, "k_T"] \arrow[d, "g"'] & T \otimes_R A \arrow[r, "\rho"] \arrow[d, "g \otimes 1"] & \mathrm{Hom}_R(A_R, N \otimes_R A_R) \arrow[dl, "\mu"] \\
N_R \arrow[r, "k_N"] & N \otimes_R A &
\end{tikzcd}$$

where $k_T(t) = t \otimes 1$, $k_N(n) = n \otimes 1$, $\rho(t \otimes a)(a') = g(t) \otimes aa'$ and $\mu(h) = h(1)$ $(t \in T,\ n \in N,\ a, a' \in A,\ h \in \mathrm{Hom}_R(A_R, N \otimes_R A_R))$. Since $\rho \circ k_T = 0$, we have $k_N \circ g = 0$. Further, since A_R is faithfully flat, k_N is a monomorphism and hence $g = 0$. So we find that $\mathrm{Hom}_R(T_R, N_R) = 0$ and $N_R \in \mathfrak{F}$.

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要　約

A を可換環 R 上の ideal algebra とする。R の Gabriel topology と A の bounded Gabriel topology は 1 対 1 に対応すること，およびこのことに関連して R の torsion theory と A の torsion theory について考察した。