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# A Note on Gabriel Topologies on Split Extensions

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Let $\mathbf{A}$ be a ring with an identity $\mathbf{1}$. $\mathbf{N}$ is called a multiplicative $\mathbf{A}$-$\mathbf{A}$-bimodule if $\mathbf{N}$ is an $\mathbf{A}$-$\mathbf{A}$-bimodule and an associative multiplication is defined such that $(\mathbf{u}+\mathbf{v})\mathbf{w}=\mathbf{uw}+\mathbf{vw}$, $\mathbf{u}(\mathbf{v}+\mathbf{w})=\mathbf{uv}+\mathbf{uw}$, $\mathbf{a}(\mathbf{uv})=(\mathbf{au})\mathbf{v}$, $(\mathbf{ua})\mathbf{v}=\mathbf{u}(\mathbf{av})$, and $(\mathbf{uv})\mathbf{a}=\mathbf{u}(\mathbf{va})$ $(\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{N}; \mathbf{a} \in \mathbf{A})$. A split extension $\mathbf{N} \rtimes \mathbf{A}$ is $\mathbf{N} \oplus \mathbf{A}$ (direct sum) as an Abelian group which has the multiplication defined by $(\mathbf{u}, \mathbf{x})(\mathbf{v}, \mathbf{y})=(\mathbf{uv}+\mathbf{uy}+\mathbf{xv}, \mathbf{xy})$ $(\mathbf{u}, \mathbf{v} \in \mathbf{N}; \mathbf{x}, \mathbf{y} \in \mathbf{A})$([2], p. 212). Then $\mathbf{N} \rtimes \mathbf{A}$ becomes a ring with an identity $(\mathbf{0}, \mathbf{1})$ and the map $\mathbf{k}: \mathbf{A} \to \mathbf{N} \rtimes \mathbf{A}$ $(\mathbf{a} \to (\mathbf{0}, \mathbf{a}))$ is a ring monomorphism. By k, $\mathbf{N} \rtimes \mathbf{A}$ is an $\mathbf{A}$-$\mathbf{A}$-bimodule. That is, $(\mathbf{u}, \mathbf{x})\mathbf{a} = (\mathbf{u}, \mathbf{x})(\mathbf{0}, \mathbf{a}) = (\mathbf{ua}, \mathbf{xa})$ and $\mathbf{a}(\mathbf{u}, \mathbf{x}) = (\mathbf{0}, \mathbf{a})(\mathbf{u}, \mathbf{x}) = (\mathbf{au}, \mathbf{ax})$. For any subset $\mathfrak{A}' \subset \mathbf{N} \rtimes \mathbf{A}$, $\mathfrak{A}' \cap \mathbf{A}$ denotes $\mathbf{k}^{-1}(\mathfrak{A}')$.

In this note, we consider the case that $\mathbf{N}$ is an ideal $(\neq \mathbf{A})$ of $\mathbf{A}$. For any right ideal $\mathfrak{A}$ and $\mathfrak{a}$ of $\mathbf{A}$ such that $\mathfrak{A} \subset \mathbf{N}$ and $\mathfrak{a}\mathbf{N} \subset \mathfrak{A}$, $(\mathfrak{A}, \mathfrak{a}) = \{(\mathbf{u}, \mathbf{a}) \mid \mathbf{u} \in \mathfrak{A} \text{ and } \mathbf{a} \in \mathfrak{a}\}$ is a right ideal of $\mathbf{N} \rtimes \mathbf{A}$. For example, for any right ideal $\mathfrak{a}$, $(\mathfrak{a} \cap \mathbf{N}, \mathfrak{a})$ and $(\mathfrak{a}\mathbf{N}, \mathfrak{a}) = \mathfrak{a}(\mathbf{N} \rtimes \mathbf{A})$ are right ideals of $\mathbf{N} \rtimes \mathbf{A}$.

For a right ideal $\mathfrak{a}$ of $\mathbf{A}$, $(\mathfrak{a} : \mathbf{a})$ denotes $\{\mathbf{x} \in \mathbf{A} \mid \mathbf{ax} \in \mathfrak{a}\}$.

A Gabriel topology $\mathfrak{F}$ on $\mathbf{A}$ is a non-empty family of right ideals satisfying :

(1) if $\mathfrak{a} \in \mathfrak{F}$, then $(\mathfrak{a} : \mathbf{a}) \in \mathfrak{F}$ for any $\mathbf{a} \in \mathbf{A}$,

(2) if $\mathfrak{b}$ is a right ideal of $\mathbf{A}$ and there exists $\mathfrak{a} \in \mathfrak{F}$ such that $(\mathfrak{b} : \mathbf{a}) \in \mathfrak{F}$ for any $\mathbf{a} \in \mathfrak{a}$, then $\mathfrak{b} \in \mathfrak{F}$ ([3], p. 12).

Let $\mathfrak{F}$ be a Gabriel topology on $\mathbf{A}$ such that $\mathbf{N} \in \mathfrak{F}$ and let $\mathfrak{F}'$ be the weakest Gabriel topology on $\mathbf{N} \rtimes \mathbf{A}$ containing $\{(\mathfrak{a}\mathbf{N}, \mathfrak{a}) \mid \mathfrak{a} \in \mathfrak{F}\}$. We put $\tilde{\mathfrak{F}} = \{\tilde{\mathbf{D}} \mid \tilde{\mathbf{D}}$ is a right ideal of $\mathbf{N} \rtimes \mathbf{A}$ such that $\tilde{\mathbf{D}} \cap \mathbf{A} \in \mathfrak{F}\}$. Clearly $\tilde{\mathfrak{F}} \supset \{(\mathfrak{A}, \mathfrak{a}) \mid \mathfrak{a} \in$ and $\mathfrak{A}$ is a right ideal of $\mathbf{A}$ such that $\mathfrak{A} \subset \mathbf{N}$ and $\mathfrak{a}\mathbf{N} \subset \mathfrak{A}\}$

Lemma 1. Under the above notations, we have $\tilde{\mathfrak{F}} \subset \mathfrak{F}'$.

Proof. Let $\tilde{\mathbf{D}} \in \tilde{\mathfrak{F}}$ and $\tilde{\mathbf{D}} \cap \mathbf{A} = \mathfrak{a} \in \mathfrak{F}$. Then $\mathfrak{a}(\mathbf{N} \rtimes \mathbf{A}) = (\mathfrak{a}\mathbf{N}, \mathfrak{a}) \subset \tilde{\mathbf{D}}$, and since $(\mathfrak{a}\mathbf{N}, \mathfrak{a}) \in \mathfrak{F}'$, we have $\tilde{\mathbf{D}} \in \mathfrak{F}'$.

Proposition 2. Under the above notations, $\tilde{\mathfrak{F}}$ is a Gabriel topology on $\mathbf{N} \rtimes \mathbf{A}$.

Proof. It is clear that for any right ideals $\tilde{\mathbf{D}}_1$ and $\tilde{\mathbf{D}}_2$, if $\tilde{\mathbf{D}}_1 \in \tilde{\mathfrak{F}}$ and $\tilde{\mathbf{D}}_1 \subset \tilde{\mathbf{D}}_2$, then

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$\tilde{\mathbf{D}}_2 \in \tilde{\mathfrak{F}}$. Let $\tilde{\mathbf{D}} \in \tilde{\mathfrak{F}}$ and $(\mathbf{u}, \mathbf{a}) \in \mathbf{N} \rtimes \mathbf{A}$. We will show that $(\tilde{\mathbf{D}} : (\mathbf{u},\mathbf{a})) \in \tilde{\mathfrak{F}}$. We put $\tilde{\mathbf{D}} \cap \mathbf{A} = \mathfrak{a} \in \mathfrak{F}$. Since $(\mathfrak{a}\mathbf{N}, \mathfrak{a}) \subset \tilde{\mathbf{D}}$, we have

$$
\begin{aligned}
(\tilde{\mathbf{D}} : (\mathbf{u},\mathbf{a})) &\supset ((\mathfrak{a}\mathbf{N}, \mathfrak{a}) : (\mathbf{u},\mathbf{a})) \\
&= \{(\mathbf{w},\mathbf{b}) \in \mathbf{N} \rtimes \mathbf{A} \mid (\mathbf{u},\mathbf{a})(\mathbf{w},\mathbf{b}) \in (\mathfrak{a}\mathbf{N}, \mathfrak{a})\} \\
&= \{(\mathbf{w},\mathbf{b}) \in \mathbf{N} \rtimes \mathbf{A} \mid (\mathbf{uw}+\mathbf{ub}+\mathbf{aw}, \mathbf{ab}) \in (\mathfrak{a}\mathbf{N}, \mathfrak{a})\} \\
&\supset ((\mathfrak{a}\mathbf{N} : \mathbf{u}+\mathbf{a}) \cap \mathbf{N}, (\mathfrak{a}\mathbf{N} : \mathbf{u}) \cap (\mathfrak{a} : \mathbf{a})) \\
&\supset (((\mathfrak{a}\mathbf{N} : \mathbf{u}) \cap (\mathfrak{a} : \mathbf{a}))\mathbf{N}, (\mathfrak{a}\mathbf{N} : \mathbf{u}) \cap (\mathfrak{a} : \mathbf{a})).
\end{aligned}
$$

Hence $(\tilde{\mathbf{D}} : (\mathbf{u},\mathbf{a})) \in \tilde{\mathfrak{F}}$. Let $\mathfrak{A}'$ be a right ideal of $\mathbf{N} \rtimes \mathbf{A}$ and there exists $\tilde{\mathbf{D}} \in \tilde{\mathfrak{F}}$ such that $(\mathfrak{A}' : (\mathbf{u},\mathbf{x})) \in \tilde{\mathfrak{F}}$ for any $(\mathbf{u},\mathbf{x}) \in \tilde{\mathbf{D}}$. We will show that $\mathfrak{A}' \in \tilde{\mathfrak{F}}$. We put $\mathfrak{a} = \tilde{\mathbf{D}} \cap \mathbf{A} \in \mathfrak{F}$ and $\mathfrak{b} = \mathfrak{A}' \cap \mathbf{A}$. For any $\mathbf{a} \in \mathfrak{a}$, since $(\mathbf{0},\mathbf{a}) \in \tilde{\mathbf{D}}$, we have $(\mathfrak{A}' : (\mathbf{0},\mathbf{a})) \cap \mathbf{A} \in \mathfrak{F}$, and we set $\mathfrak{c} = (\mathfrak{A}' : (\mathbf{0},\mathbf{a})) \cap \mathbf{A}$. Since $(\mathbf{a}\mathfrak{c}\mathbf{N}, \mathbf{a}\mathfrak{c}) = (\mathbf{0},\mathbf{a})(\mathfrak{c}\mathbf{N}, \mathfrak{c}) \subset \mathfrak{A}'$, it follows $\mathbf{a}\mathfrak{c} = (\mathbf{a}\mathfrak{c}\mathbf{N}, \mathbf{a}\mathfrak{c}) \cap \mathbf{A} \subset \mathfrak{A}' \cap \mathbf{A} = \mathfrak{b}$. Hence $\mathfrak{c} \subset (\mathfrak{b} : \mathbf{a})$ and $(\mathfrak{b} : \mathbf{a}) \in \mathfrak{F}$. Since $\mathfrak{F}$ is a Gabriel topology, we have $\mathfrak{b} \in \mathfrak{F}$ and $\mathfrak{A}' \in \tilde{\mathfrak{F}}$.

Corollary 3. We have $\tilde{\mathfrak{F}} = \mathfrak{F}'$, and $\mathfrak{F}$ is $\mathbf{N} \rtimes \mathbf{A}$-good in the sense of [1] (p. 517).

Lemma 4. Let $\mathfrak{F}'$ be a Gabriel topology on $\mathbf{N} \rtimes \mathbf{A}$. Then $\mathfrak{F} = \{\mathfrak{a} \mid \mathfrak{a}$ is a right ideal of $\mathbf{A}$ such that $(\mathbf{N}, \mathfrak{a}) \in \mathfrak{F}'\}$ is a Gabriel topology on $\mathbf{A}$.

Proof. Let $\mathfrak{a} \in \mathfrak{F}$ and $\mathbf{a} \in \mathbf{A}$. Since $(\mathbf{N}, \mathfrak{a}) \in \mathfrak{F}'$ and $(\mathbf{N}, (\mathfrak{a} : \mathbf{a})) = ((\mathbf{N}, \mathfrak{a}) : (\mathbf{0}, \mathbf{a})) \in \mathfrak{F}'$, we have $(\mathfrak{a} : \mathbf{a}) \in \mathfrak{F}$. Next, let $\mathfrak{b}$ be a right ideal of $\mathbf{A}$ and assume that there exists $\mathfrak{a} \in \mathfrak{F}$ such that for any $\mathbf{a} \in \mathfrak{a}$, $(\mathfrak{b} : \mathbf{a}) \in \mathfrak{F}$. Then for any $\mathbf{u} \in \mathbf{N}$ and $\mathbf{a} \in \mathfrak{a}$, since $((\mathbf{N}, \mathfrak{b}) : (\mathbf{u},\mathbf{a})) = (\mathbf{N}, (\mathfrak{b} : \mathbf{a})) \in \mathfrak{F}'$, we have $(\mathbf{N}, \mathfrak{b}) \in \mathfrak{F}'$ and $\mathfrak{b} \in \mathfrak{F}$.

Lemma 5. Let $\mathfrak{F}'$ be a Gabriel topology on $\mathbf{N} \rtimes \mathbf{A}$ such that for any $\mathfrak{A}' \in \mathfrak{F}'$, $(\mathfrak{A}' \cap \mathbf{A})(\mathbf{N} \rtimes \mathbf{A}) \in \mathfrak{F}'$. Then $\{\mathfrak{A}' \cap \mathbf{A} \mid \mathfrak{A}' \in \mathfrak{F}'\} = \mathfrak{F}$ where $\mathfrak{F}$ is that in Lemma 4.

Proof. Let $\mathfrak{a} \in \mathfrak{F}$. Since $(\mathbf{N}, \mathfrak{a}) \in \mathfrak{F}'$ and $(\mathbf{N}, \mathfrak{a}) \cap \mathbf{A} = \mathfrak{a}$, we have $\mathfrak{a} \in \{\mathfrak{A}' \cap \mathbf{A} \mid \mathfrak{A}' \in \mathfrak{F}'\}$. Conversely for $\mathfrak{A}' \in \mathfrak{F}'$, we put $\mathfrak{A}' \cap \mathbf{A} = \mathfrak{a}$. Then by the assumptions of $\mathfrak{F}'$, $(\mathfrak{a}\mathbf{N}, \mathfrak{a}) \in \mathfrak{F}'$ and hence $(\mathbf{N}, \mathfrak{a}) \in \mathfrak{F}'$.

Theorem 6. Let $\mathbf{N}$ be an ideal on $\mathbf{A}$. Then there is a one to one correspondence between Gabriel topologies $\mathfrak{F}$ on $\mathbf{A}$ such that $\mathbf{N} \in \mathfrak{F}$ and Gabriel topologies $\mathfrak{F}'$ on $\mathbf{N} \rtimes \mathbf{A}$ such that $(\mathbf{N}, \mathbf{N}) \in \mathfrak{F}'$ and for any $\mathfrak{A}' \in \mathfrak{F}'$, $(\mathfrak{A}' \cap \mathbf{A})(\mathbf{N} \rtimes \mathbf{A}) \in \mathfrak{F}'$.

Let $\mathfrak{F}$ be the Gabriel topology for the Lambek torsion theory. Then $\mathfrak{a} \in \mathfrak{F}$ if and only if for any $\mathbf{a} \in \mathbf{A}$ and $\mathbf{0} \neq \mathbf{b} \in \mathbf{A}$, there exists $\mathbf{r} \in \mathbf{A}$ such that $\mathbf{ar} \in \mathfrak{a}$ and $\mathbf{br} \neq \mathbf{0}$ ([3], p. 17).

Proposition 7. Under the notations in Theorem 6, if $\mathfrak{F}$ is the Gabriel topology for the Lambek torsion theory of $\mathbf{A}$, then $\mathfrak{F}'$ is also the Gabriel topology for the Lambek torsion theory of $\mathbf{N} \rtimes \mathbf{A}$.

Proof. Let $\tilde{\mathbf{D}} \in \mathfrak{F}'$ and $\mathfrak{a} = \tilde{\mathbf{D}} \cap \mathbf{A} \in \mathfrak{F}$. Then $(\mathfrak{a}\mathbf{N}, \mathfrak{a}) \subset \tilde{\mathbf{D}}$. Let $(\mathbf{u}, \mathbf{a}) \in \mathbf{N} \ltimes \mathbf{A}$ and $\mathbf{0} \neq (\mathbf{v}, \mathbf{b}) \in \mathbf{N} \ltimes \mathbf{A}$. If $\mathbf{b} \neq \mathbf{0}$, since $\mathfrak{a}\mathbf{N} \in \mathfrak{F}$, there exists $\mathbf{r} \in \mathbf{A}$ such that $\mathbf{ur} \in \mathfrak{a}\mathbf{N}$ and $\mathbf{br} \neq \mathbf{0}$. Then, there exists $\mathbf{s} \in \mathbf{A}$ such that $\mathbf{ars} \in \mathfrak{a}$ and $\mathbf{brs} \neq \mathbf{0}$. Hence $(\mathbf{u}, \mathbf{a})(\mathbf{0}, \mathbf{rs}) = (\mathbf{urs}, \mathbf{ars}) \in (\mathfrak{a}\mathbf{N}, \mathfrak{a}) \subset \tilde{\mathbf{D}}$ and $(\mathbf{v}, \mathbf{b})(\mathbf{0}, \mathbf{rs}) = (\mathbf{vrs}, \mathbf{brs}) \neq \mathbf{0}$. If $\mathbf{v} \neq 0$, there exists $\mathbf{r} \in \mathbf{A}$ such that $\mathbf{ar} \in \mathfrak{a}$ and $\mathbf{vr} \neq \mathbf{0}$. Then, there exists $\mathbf{s} \in \mathbf{A}$ such that $\mathbf{urs} \in \mathfrak{a}\mathbf{N}$ and $\mathbf{vrs} \neq \mathbf{0}$. Hence $(\mathbf{u}, \mathbf{a})(\mathbf{0}, \mathbf{rs}) = (\mathbf{urs}, \mathbf{ars}) \in (\mathfrak{a}\mathbf{N}, \mathfrak{a}) \subset \tilde{\mathbf{D}}$ and $(\mathbf{v}, \mathbf{b})(\mathbf{0}, \mathbf{rs}) = (\mathbf{vrs}, \mathbf{brs}) \neq \mathbf{0}$.

Proposition 8. Under the notations in Theorem 6, $\mathbf{t}(\mathbf{A}) = \mathbf{0}$ if and only if $\mathbf{t}'(\mathbf{N} \ltimes \mathbf{A}) = \mathbf{0}$, where $\mathbf{t}(\mathbf{A})$ (resp. $\mathbf{t}'(\mathbf{N} \ltimes \mathbf{A})$) is the torsion submodule of $\mathbf{A}$ (resp. $\mathbf{N} \ltimes \mathbf{A}$) associated to $\mathfrak{F}$ (resp. $\mathfrak{F}'$).

Proof. Let $\mathbf{t}(\mathbf{A}) = \mathbf{0}$. By definitions,

$$\begin{aligned} \mathbf{t}'(\mathbf{N} \ltimes \mathbf{A}) &= \{(\mathbf{u}, \mathbf{x}) \in \mathbf{N} \ltimes \mathbf{A} \mid (\mathbf{0} : (\mathbf{u}, \mathbf{x})) \in \mathfrak{F}'\} \\ &= \{(\mathbf{u}, \mathbf{x}) \in \mathbf{N} \ltimes \mathbf{A} \mid (\mathbf{0} : (\mathbf{u}, \mathbf{x})) \cap \mathbf{A} \in \mathfrak{F}\} \end{aligned}$$

Hence for any $(\mathbf{u}, \mathbf{x}) \in \mathbf{t}'(\mathbf{N} \ltimes \mathbf{A})$, since

$$\begin{aligned} (\mathbf{0} : (\mathbf{u}, \mathbf{x})) \cap \mathbf{A} &= \{\mathbf{a} \in \mathbf{A} \mid (\mathbf{ua}, \mathbf{xa}) = \mathbf{0}\} \\ &= (\mathbf{0} : \mathbf{u}) \cap (\mathbf{0} : \mathbf{x}), \end{aligned}$$

it follows $(\mathbf{0} : \mathbf{u}) \in \mathfrak{F}$, $(\mathbf{0} : \mathbf{x}) \in \mathfrak{F}$ and $\mathbf{u} = \mathbf{x} = \mathbf{0}$. Next, let $\mathbf{t}'(\mathbf{N} \ltimes \mathbf{A}) = \mathbf{0}$.
For any $\mathbf{x} \in \mathbf{t}(\mathbf{A})$, since

$$\begin{aligned} (\mathbf{0} : (\mathbf{0}, \mathbf{x})) &= \{(\mathbf{u}, \mathbf{a}) \in \mathbf{N} \ltimes \mathbf{A} \mid (\mathbf{xu}, \mathbf{xa}) = \mathbf{0}\} \\ &= ((\mathbf{0} : \mathbf{x}) \cap \mathbf{N}, (\mathbf{0} : \mathbf{x})) \in \mathfrak{F}', \end{aligned}$$

we have $(\mathbf{0}, \mathbf{x}) \in \mathbf{t}'(\mathbf{N} \ltimes \mathbf{A})$ and $\mathbf{x} = \mathbf{0}$.

## References


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## 要　約

Aを環，Nを ideal N⋉Aを split extension とする。A の Gabriel topology $\mathfrak{F}$ で N∈$\mathfrak{F}$ であるものと N⋉A の Gabriel topology $\mathfrak{F}'$ で (N, N)∈$\mathfrak{F}'$ かつ任意の $\mathfrak{A}'$∈$\mathfrak{F}'$ に対して ($\mathfrak{A}'$∩A)(N⋉A)∈$\mathfrak{F}'$ を満たすものは 1 対 1 に対応することを示した。