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# Supplement to Note on Gabriel Topologies on Split Extensions

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This note is a continuation of the previous note[2]. Here, we consider the case where $\mathbf{A}$ is a commutative Noetherian ring.

An $\mathbf{A}$-module $\mathbf{M}$ is said to be finitely presented if $\mathbf{M}$ is isomorphic to $\mathbf{F}/\mathbf{K}$, where $\mathbf{F}$ is finitely generated and free and $\mathbf{K}$ is finitely generated ([1], p.24). If $\mathbf{A}$ is Noetherian, every finitely generated module is finitely presented.

Lemma 1 ([1] p.26). Let $\mathbf{A}$ be a commutative ring and $\mathbf{B}$ be an $\mathbf{A}$-algebra which is flat as an $\mathbf{A}$-module. Then for any finitely presented $\mathbf{A}$-module $\mathbf{M}$ and any $\mathbf{A}$-module $\mathbf{X}$,

$$\Phi\colon \mathbf{Hom}_A(\mathbf{M},\mathbf{X})\otimes_A\mathbf{B} \to \mathbf{Hom}_B(\mathbf{M}\otimes_A\mathbf{B},\mathbf{X}\otimes_A\mathbf{B})$$
$$(\mathbf{f}\otimes\mathbf{b} \to (\mathbf{m}\otimes\mathbf{z} \to \mathbf{f}(\mathbf{m})\otimes\mathbf{bz}))$$

is an isomorphism.

For a Gabriel topology $\mathfrak{F}$ on $\mathbf{A}$, $\mathbf{A}$ is said to be $\mathfrak{F}$-injective, if for any $\mathfrak{a}\in\mathfrak{F}$,

$\alpha\colon \mathbf{A} \to \mathbf{Hom}_A(\mathfrak{a},\mathbf{A})$ $(\mathbf{a} \to (\mathbf{y} \to \mathbf{ay}))$ is surjective ([3] p.29).

Proposition 2. Let $\mathbf{A}$ be a commutative Noetherian ring and $\mathfrak{F}$ be a Gabriel topology such that $\mathbf{N}\in\mathfrak{F}$ and $\mathbf{N}_A$ is flat. Let $\mathfrak{F}'$ be the Gabriel topology on $\mathbf{N}\ltimes\mathbf{A}$ corresponding to $\mathfrak{F}$ in [2 Theorem 6]. Then,

(1) if $\mathbf{N}\ltimes\mathbf{A}$ is $\mathfrak{F}'$-injective, then $\mathbf{A}$ is also $\mathfrak{F}$-injective,

(2) if $\mathbf{A}$ is $\mathfrak{F}$-torsion free (i.e. $\mathbf{t}(\mathbf{A})=\mathbf{0}$) and $\mathbf{A}$ is $\mathfrak{F}$-injective, then $\mathbf{N}\ltimes\mathbf{A}$ is $\mathfrak{F}'$-injective.

Proof. (1) We put $\mathbf{B}=\mathbf{N}\ltimes\mathbf{A}$. For any $\mathfrak{a}\in\mathfrak{F}$, the sequence

$$\mathbf{A} \xrightarrow{\alpha} \mathbf{Hom}_A(\mathfrak{a},\mathbf{A}) \to \mathbf{Hom}_A(\mathfrak{a},\mathbf{A})/\mathbf{Im}(\alpha) \to \mathbf{0}$$

is exact. Since ${}_A\mathbf{B}$ is flat,

$$\mathbf{B} \xrightarrow{\alpha\otimes 1} \mathbf{Hom}_A(\mathfrak{a},\mathbf{A})\otimes_A\mathbf{B} \to \mathbf{Hom}_A(\mathfrak{a},\mathbf{A})/\mathbf{Im}(\alpha)\otimes_A\mathbf{B} \to\mathbf{0}$$

is exact and by Lemma 1 and assumptions, $\alpha\otimes 1$ is surjective. Hence,
$\mathbf{Hom}_A(\mathfrak{a},\mathbf{A})/\mathbf{Im}(\alpha)\otimes_A\mathbf{B}=\mathbf{0}$ and
$\mathbf{Hom}_A(\mathfrak{a},\mathbf{A})/\mathbf{Im}(\alpha)=\mathbf{0}$.

(2) Let $\mathbf{I}'\in\mathfrak{F}'$. Then $\mathfrak{a}=\mathbf{I}'\cap\mathbf{A}\in\mathfrak{F}$ and $\mathfrak{a}\mathbf{B}=(\mathfrak{a}\mathbf{N},\mathfrak{a})\in\mathfrak{F}'$, and the following diagram is commutative:

$$\begin{tikzcd}
\mathbf{Hom}_B(\mathbf{I}',\mathbf{B}) \arrow[r, "\lambda"] & \mathbf{Hom}_B(\mathfrak{a}\mathbf{B},\mathbf{B}) \arrow[r, "\mu"] & \mathbf{Hom}_B(\mathfrak{a}\otimes_A\mathbf{B},\mathbf{B}) \\
\mathbf{B} \arrow[u, "\beta"] \arrow[rr, "\alpha\otimes 1"] & & \mathbf{Hom}_A(\mathfrak{a},\mathbf{A})\otimes_A\mathbf{B} \arrow[u, "\Phi"']
\end{tikzcd}$$

where $[\beta(\mathbf{b})](\mathbf{u})=\mathbf{bu}$, $\lambda(\mathbf{g})=\mathbf{g}\mid(\mathfrak{a}\mathbf{B})$, $[\mu(\mathbf{h})](\mathbf{a}\otimes\mathbf{b})=\mathbf{h}(\mathbf{ab})$ and $\Phi$ is the isomorphism defined in Lemma 1. Hence, for any $\mathbf{g}\in\mathbf{Hom}_B(\mathbf{I}',\mathbf{B})$, there exists $\mathbf{b}_0\in\mathbf{B}$ such that $\mathbf{g}\mid(\mathfrak{a}\mathbf{B})=(\lambda\circ\beta)(\mathbf{b}_0)=[\mu^{-1}\circ\Phi^{-1}\circ(\alpha\otimes 1)](\mathbf{b}_0)$. By [2] Proposition 8, $\mathbf{B}$ is $\mathfrak{F}'$-torsion free and $\mathbf{I}'/\mathfrak{a}\mathbf{B}$ is $\mathfrak{F}'$-torsion, we have $\mathbf{Hom}_B(\mathbf{I}'/\mathfrak{a}\mathbf{B},\mathbf{B})=\mathbf{0}$ and $\mathbf{g}(\mathbf{u})=\mathbf{b}_0\mathbf{u}$.

$\mathbf{A}$ is said to be $\mathfrak{F}$-closed if $\mathbf{A}$ is $\mathfrak{F}$-torsion free and $\mathfrak{F}$-injective ([3] p.37).

Corollary 3. Under the assumptions of Proposition 2, $\mathbf{A}$ is $\mathfrak{F}$-closed if and only if $\mathbf{N}\ltimes\mathbf{A}$ is $\mathfrak{F}'$-closed.

## References

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