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酪農学園大学紀要　別　刷　第 31 巻　第 1 号

Reprinted from

"Journal of Rakuno Gakuen University" Vol.31, No.1 (2006)

J. Rakuno Gakuen Univ., **31** (1) : 1~6 (2006)

A note on Morita modules and quotient rings

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(June 2006)

In this note, every ring has an identity 1 and every module over a ring is unital. A ring extension A/B means B is a subring of A containing 1_A (the identity of A) and a ring homomorphism means such one that the image of 1 is 1.

A homomorphism will be usually written at the opposite side of the scalar.

An A′-A-bimodule ${}_{A'}U_A$ is said to be a Morita module if U_A is a progenerator and $A' = \mathrm{End}(U_A)$ ([7] p.98).

We consider ring extensions A/B and A′/B′ which have Morita modules ${}_{A'}U_A$ and ${}_{B'}V_B$.

Lemma 1. ([7], p.111). Under the above situations, the following statements are equivalent.

(1) There exists a B′-A-isomorphism ϕ: ${}_{B'}V \otimes {}_B A_A \to {}_{B'}U_A$.

(2) There exists an A′-B-isomorphism ϕ': ${}_{A'}A' \otimes {}_{B'}V_B \to {}_{A'}U_B$.

In this case, the equation $\phi(v \otimes 1) = \phi'(1 \otimes v)$ holds for any $v \in V$.

Proof. (1)⇒(2): Since ${}_{B'}V$ is finitely generated and projective, we have

$$\begin{aligned} &{}_{A'}A' \otimes {}_{B'}V_B = {}_{A'}\mathrm{Hom}(U_A,U_A) \otimes {}_{B'}V_B \cong {}_{A'}\mathrm{Hom}(\mathrm{Hom}({}_{B'}V,{}_{B'}U)_A,U_A)_B \\ &\cong {}_{A'}\mathrm{Hom}(\mathrm{Hom}({}_{B'}V,{}_{B'}V \otimes {}_B A)_A,U_A)_B \cong {}_{A'}\mathrm{Hom}(\mathrm{Hom}({}_{B'}V,{}_{B'}V) \otimes {}_B A_A,U_A)_B \\ &\cong {}_{A'}\mathrm{Hom}(A_A,U_A)_B \cong {}_{A'}U_B. \end{aligned}$$

Hence an isomorphism ϕ' exists and the above correspondences are given by

$$\begin{aligned} &\Sigma a'_j \otimes v_j \to \Sigma(x \to a'_j\ x) \otimes v_j \to (g \to \Sigma a'_j \cdot (v_j)g) \\ &\to (h \to \Sigma a'_j \cdot \phi((v_j)h)) = ((v \to v \otimes a) \to ua) \leftarrow (1 \otimes a \to ua) \\ &\leftarrow (a \to ua) \leftarrow u = \phi'(\Sigma a'_j \otimes v_j). \end{aligned}$$

So, for any $a \in A$, $\Sigma a'_j \cdot \phi(v_j \otimes 1) \cdot a = \Sigma a'_j \cdot \phi(v_j \otimes a) = ua = \phi'(\Sigma a'_j \otimes v_j) \cdot a = \Sigma a'_j \cdot \phi'(1 \otimes v_j) \cdot a$.

Especially, $\phi(v \otimes 1) = \phi'(1 \otimes v)$ for any $v \in V$.

(2)⇒(1): This is similarly proved.

If the conditions of Lemma 1 are satisfied, then ring extensions A/B and A′/B′ are said to be Morita equivalent ([7] p.111, [5] p.74).

Proposition 2. If Lemma 1 is satisfied, then the following statements are equivalent.

(1) The map ${}_A A \otimes {}_B A_A \to {}_A A_A$ $(x \otimes y \to xy)$ is an isomorphism.

(2) The map ${}_{A'}A' \otimes {}_{B'}V \otimes {}_B A_A \to {}_{A'}U_A$ $(a' \otimes v \otimes a \to a'\phi(v \otimes a) = \phi'(a' \otimes v)a)$ is an isomorphism.

(3) The map ${}_{A'}A' \otimes {}_{B'}A'_{A'} \to {}_{A'}A'_{A'}$ $(x' \otimes y' \to x'y')$ is an isomorphism.

Proof. (1)⇒(2): By Lemma 1, we have

$$\begin{aligned} &{}_{A'}A' \otimes {}_{B'}V \otimes {}_B A_A \cong {}_{A'}U \otimes {}_B A_A \cong {}_{A'}U \otimes {}_A A \otimes {}_B A_A \cong {}_{A'}U \otimes {}_A A_A \cong {}_{A'}U_A. \\ &a' \otimes v \otimes a \to \phi'(a' \otimes v) \otimes a \to \phi'(a' \otimes v) \otimes 1 \otimes a \to \phi'(a' \otimes v) \otimes a \to \phi'(a' \otimes v)a \\ &\phi'^{-1}(u) \otimes 1 \leftarrow u \otimes 1 \leftarrow u \otimes 1 \otimes 1 \leftarrow u \otimes 1 \leftarrow u. \end{aligned}$$

(2)⇒(3): Since ${}_A U_A \cong {}_{A'}A' \otimes {}_{B'}V \otimes {}_B A_A \cong {}_{A'}A' \otimes {}_{B'}U_A \cong {}_{A'}A' \otimes {}_{B'}A' \otimes {}_{A'}U_A$

$$u \to \phi'^{-1}(u) \otimes 1 \to \Sigma a'_j \otimes \phi(v_j \otimes 1) \to \Sigma a'_j \otimes 1 \otimes \phi(v_j \otimes 1) \text{ where } \phi'^{-1}(u) = \Sigma a'_j \otimes v_j$$

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$\sum\phi'(x'\otimes w_k)s_k \leftarrow x'\otimes\phi^{-1}(y't) \leftarrow x'\otimes y't \leftarrow x'\otimes y'\otimes t$ where $\phi^{-1}(y't)=\sum w_k\otimes s_k$
and U_A is finitely generated and projective, we have

$$_{A'}A'\otimes_{B'}A'_{A'} \cong {}_{A'}A'\otimes_{B'}A'\otimes_{A'}\mathrm{Hom}(U_A,U_A)_{A'} \cong {}_{A'}\mathrm{Hom}(U_A,A'\otimes_{B'}A'\otimes_{A'}U_A)_{A'}$$
$$\cong {}_{A'}\mathrm{Hom}(U_A,U_A)_{A'} \cong {}_{A'}A'_{A'}.$$
$$x'\otimes y' \to x'\otimes 1\otimes(u\to y'u) \to (t\to x'\otimes 1\otimes y't = x'\otimes y'\otimes t)$$
$$\to (t\to\sum\phi'(x'\otimes w_k)s_k = \sum x'\phi(w_k\otimes s_k) = x'y't) \to x'y'.$$

(3)⇒(2): By Lemma 1, we have $_{A'}A'\otimes_{B'}V\otimes_B A_A \cong {}_{A'}A'\otimes_{B'}U_A \cong {}_{A'}A'\otimes_{B'}A'\otimes_{A'}U_A$
$\cong {}_{A'}U_A$ $\quad a'\otimes v\otimes a \to a'\otimes\phi(v\otimes a)\to a'\otimes 1\otimes\phi(v\otimes a)\to a'\otimes\phi(v\otimes a).$
(2)⇒(1): Since $_{A'}U_A \cong {}_{A'}A'\otimes_{B'}V\otimes_B A_A \cong {}_{A'}U\otimes_B A_A \cong {}_{A'}U\otimes_A A\otimes_B A_A$

$$u\to\phi'^{-1}(u)\otimes 1\to u\otimes 1\to u\otimes 1\otimes 1$$

and $_{A'}U$ is finitely generated and projective, we have

$$_A A\otimes_B A_A \cong {}_A\mathrm{Hom}(_{A'}U,_{A'}U)\otimes_A A\otimes_B A_A \cong {}_A\mathrm{Hom}(_{A'}U,_{A'}U\otimes_A A\otimes_B A_A)$$
$$\cong {}_A\mathrm{Hom}(_{A'}U,_{A'}U)_A \cong {}_A A_A$$
$$x\otimes y\to(u\to ux)\otimes 1\otimes y\to(u\to ux\otimes 1\otimes y)\to(u\to uxy)\to xy.$$

Lemma 3. If Lemma 1 is satisfied, then
(1) $_A\mathrm{Hom}(U_A,A_A)_{A'} \cong {}_A\mathrm{Hom}(_{A'}U,_{A'}A')_{A'}$ $(f\to f')$
(2) $_B\mathrm{Hom}(V_B,B_B)_{B'} \cong {}_B\mathrm{Hom}(_{B'}V,_{B'}B')_{B'}$ $(g\to g')$
For the above correspondences, the equations $y\cdot f(x) = (y)f'\cdot x$ and $z\cdot g(w) = (z)g'\cdot w$ hold for any $x,y\in U$ and $w,z\in V$.

Proof. (1) We have $_A\mathrm{Hom}(U_A,A_A)_{A'} \cong {}_A\mathrm{Hom}(U_A,\mathrm{Hom}(_{A'}U,_{A'}U)_A)_{A'}$

$$\cong {}_A\mathrm{Hom}(_{A'}U,_{A'}\mathrm{Hom}(U_A,U_A))_{A'} \cong {}_A\mathrm{Hom}(_{A'}U,_{A'}A')_{A'}$$
$$f\to h_1:(x_1\to(y_1\to y_1\cdot f(x_1))$$
$$\to h_2:(y_2\to(x_2\to(y_2)[h_1(x_2)] = y_2\cdot f(x_2))\to f'$$

and $(y_2)f'\cdot x_2 = [(y_2)f'](x_2) = y_2\cdot f(x_2)$.
(2) is similarly proved.

The modules which were defined in (1) and (2) of Lemma 3 will be written as U^* and V^* respectively.

Lemma 4. If Lemma 1 is satisfied, then
(1) $_A U^*_{B'} \cong {}_A\mathrm{Hom}(V_B,A_B)_{B'}$
(2) $_B U^*_{A'} \cong {}_B\mathrm{Hom}(_{B'}V,_{B'}A')_{A'}$
(3) $_{A'}U\otimes_B V^*_{B'} \cong {}_{A'}A'_{B'}$
(4) $_{B'}V\otimes_B U^*_{A'} \cong {}_{B'}A'_{A'}$
(5) $_B V^*\otimes_{B'}U_A \cong {}_B A_A$
(6) $_A U^*\otimes_{B'}V_B \cong {}_A A_B$

Proof. (1) By Lemma 1, we have $_A U^*_{B'} \cong {}_A\mathrm{Hom}(U_A,A_A)_{B'} \cong {}_A\mathrm{Hom}(V\otimes_B A_A,A_A)_{B'}$
$\cong {}_A\mathrm{Hom}(V_B,\mathrm{Hom}(A_A,A_A)_B)_{B'} \cong {}_A\mathrm{Hom}(V_B,A_B)_{B'}$.
(2) By Lemma 1, we have $_B U^*_{A'} \cong {}_B\mathrm{Hom}(_{A'}U,_{A'}A')_{A'} \cong {}_B\mathrm{Hom}(_{A'}A'\otimes_{B'}V,_{A'}A')_{A'}$
$\cong {}_B\mathrm{Hom}(_{B'}V,_{B'}\mathrm{Hom}(_{A'}A',_{A'}A'))_{A'} \cong {}_B\mathrm{Hom}(_{B'}V,_{B'}A')_{A'}$.
(3) Since V_B is finitely generated and projective, by Lemma 1, we have
$_{A'}U\otimes_B V^*_{B'} \cong {}_{A'}U\otimes_B\mathrm{Hom}(V_B,B_B)_{B'} \cong {}_{A'}\mathrm{Hom}(V_B,U\otimes_B B_B)_{B'} \cong {}_{A'}\mathrm{Hom}(V_B,U_B)_{B'}$
$\cong {}_{A'}\mathrm{Hom}(V_B,A'\otimes_{B'}V_B)_{B'} \cong {}_{A'}A'\otimes_{B'}\mathrm{Hom}(V_B,V_B)_{B'} \cong {}_{A'}A'_{B'}$.
(4) Since U_A is finitely generated and projective, by Lemma 1, we have
$_{B'}V\otimes_B U^*_{A'} \cong {}_{B'}V\otimes_B\mathrm{Hom}(U_A,A_A)_{A'} \cong {}_{B'}\mathrm{Hom}(U_A,V\otimes_B A_A)_{A'} \cong {}_{B'}\mathrm{Hom}(U_A,U_A)_{A'} \cong {}_{B'}A'_{A'}$.
(5) By Lemma 1, we have $_B V^*\otimes_{B'}U_A \cong {}_B V^*\otimes_{B'}V\otimes_B A_A \cong {}_B A_A$.
(6) Since $_{A'}U$ is finitely generated and projective, by Lemma 1, we have
$_A U^*\otimes_{B'}V_B \cong {}_A\mathrm{Hom}(_{A'}U,_{A'}A')\otimes_{B'}V_B \cong {}_A\mathrm{Hom}(_{A'}U,_{A'}A'\otimes_{B'}V)_B \cong {}_A\mathrm{Hom}(_{A'}U,_{A'}U)_B \cong {}_A A_B$.

Proposition 5. If Lemma 1 is satisfied, then we have

(1) ${}_{B}A$ is flat if and only if ${}_{B'}A'$ is flat.

(2) A_B is flat if and only if $A'_{B'}$ is flat.

Proof. (1) If ${}_{B}A$ is flat, then ${}_{B'}V \otimes_B A \cong {}_{B'}U$ is flat and hence ${}_{B'}U \otimes_A U^* \cong {}_{B'}A'$ is flat. If ${}_{B'}A'$ is flat, then ${}_{B'}U \otimes_A U^*$ is flat and by Lemma 4 (5), ${}_{B}V^* \otimes_{B'} U \otimes_A U^* \cong {}_{B}A \otimes_A U^* \cong {}_{B}U^*$ is flat. Hence ${}_{B}U^* \otimes_{A'} U \cong {}_{B}A$ is flat.

(2) is similarly proved by Lemma 4 (6).

Let ${}_{A}\boldsymbol{M}_{B}$ be the isomorphism classes of A-B-bimodules. We consider the following two diagrams.

(diagram 1)

$$\begin{tikzcd}
 & {}_{A'}\boldsymbol{M}_{B} \arrow[dl, shift left, "{[U^*_{A'}\otimes]}"] \arrow[dr, shift left, "{[\otimes_B V^*]}"] & \\
{}_{A}\boldsymbol{M}_{B} \arrow[ur, shift left, "{[U_A\otimes]}"] \arrow[dr, shift left, "{[\otimes_B V^*]}"] & & {}_{A'}\boldsymbol{M}_{B'} \arrow[ul, shift left, "{[\otimes_{B'} V]}"] \arrow[dl, shift left, "{[U^*_{A'}\otimes]}"] \\
 & {}_{A}\boldsymbol{M}_{B'} \arrow[ul, shift left, "{[\otimes_{B'} V]}"] \arrow[ur, shift left, "{[U_A\otimes]}"] &
\end{tikzcd}$$

(diagram 2)

$$\begin{tikzcd}
 & {}_{B}\boldsymbol{M}_{A'} \arrow[dl, shift left, "{[\otimes_{A'} U]}"] \arrow[dr, shift left, "{[V_B\otimes]}"] & \\
{}_{B}\boldsymbol{M}_{A} \arrow[ur, shift left, "{[\otimes_A U^*]}"] \arrow[dr, shift left, "{[V_B\otimes]}"] & & {}_{B'}\boldsymbol{M}_{A'} \arrow[ul, shift left, "{[V^*_{B'}\otimes]}"] \arrow[dl, shift left, "{[\otimes_{A'} U]}"] \\
 & {}_{B'}\boldsymbol{M}_{A} \arrow[ul, shift left, "{[V^*_{B'}\otimes]}"] \arrow[ur, shift left, "{[\otimes_A U^*]}"] &
\end{tikzcd}$$

where, for example, the map $[U_A \otimes\ ]$ is defined by $[U_A \otimes\ ][{}_{A}X_{B}] = [{}_{A'}U \otimes_A X_B]$.

If the conditions of Lemma 1 and Proposition 2 are satisfied, in each isomorphism classes of diagram 1 and diagram 2, a binary multiplication are defined with a left identity and a right identity respectively.

For example, the multiplication in ${}_{A}\boldsymbol{M}_{B}$ is defined by $[{}_{A}X_{B}][{}_{A}Y_{B}] = [{}_{A}X \otimes_B Y_B]$, and since ${}_{A}A \otimes_B Y_B \cong {}_{A}A \otimes_B A \otimes_A Y_B \cong {}_{A}A \otimes_A Y_B \cong {}_{A}Y_B$, $[{}_{A}A_{B}]$. is a left identity. The multiplication in ${}_{A'}\boldsymbol{M}_{B}$ is defined by $[{}_{A'}X_{B}][{}_{A'}Y_{B}] = [{}_{A'}U \otimes_A U^* \otimes_{A'} X \otimes_B U^* \otimes_{A'} Y_B] = [{}_{A'}X \otimes_B U^* \otimes_{A'} Y_B]$ and $[{}_{A'}U_{B}]$. is a left identity. By Lemma 4 (4), the multiplication in ${}_{A'}\boldsymbol{M}_{B'}$ is defined by $[{}_{A'}X_{B'}][{}_{A'}Y_{B'}] = [{}_{A'}U \otimes_A U^* \otimes_{A'} X \otimes_{B'} V \otimes_B U^* \otimes_{A'} Y \otimes_{B'} V \otimes_B V^*{}_{B'}] = [{}_{A'}X \otimes_{B'} A' \otimes_{A'} Y_B] = [{}_{A'}X \otimes_{B'} Y_{B'}]$ and $[{}_{A'}A'_{B'}]$ is a left identity. The multiplication in ${}_{A}\boldsymbol{M}_{B'}$ is defined by $[{}_{A}X_{B'}][{}_{A}Y_{B'}] = [{}_{A}X \otimes_{B'} V \otimes_B Y_{B'}]$ and by Lemma 4 (1), $[{}_{A}U^*{}_{B'}] = [{}_{A}A \otimes_B V^*{}_{B'}]$ is a left identity.

We can consider similarly in the diagram 2, and $[{}_{B}A_{A}]$ is a right identity of ${}_{B}\boldsymbol{M}_{A}$.

Proposition 6. Maps of diagram 1 (resp. diagram 2) are bijective and preserve the multiplications and left (resp. right) identities.

Let ${}_{B'}V_B$ be a Morita module. It is well known that there exists a one-to-one correspondence between the set of (two sided) ideals $\{J'\}$of B' and the set of (two sided) ideals $\{J\}$of B under the correspondence
$J' \to \{b \in B \mid Vb \subseteq J'V\}$ and $\{b' \in B' \mid b'V \subseteq VJ\} \leftarrow J$ ([2], p.6).

In this note, we will always assume that B is a right Noetherian and right hereditary ring. That is, B is right Noetherian and every right ideal of B is projective. Then B' is also a right Noetherian and right hereditary ring ([3], p.378).

An ideal J of B is called dense as a right ideal if $bJ = 0$ then $b = 0$ for any $b \in B$ ([6], p.96).

Let $\boldsymbol{D} = \{J \mid J$ is an ideal of B and dense as a right ideal of B$\}$. Then,

(1) for ideals J and K of B, if $J \in \boldsymbol{D}$ and $J \subseteq K$, then $K \in \boldsymbol{D}$,

(2) for ideals J and K of B, if $J \in \boldsymbol{D}$ and $K \in \boldsymbol{D}$, then $JK \in \boldsymbol{D}$.

That is, $\boldsymbol{D}$ is a filter.

Let $A = \varinjlim_{\boldsymbol{D}} \mathrm{Hom}(J_B, B_B)$ (a ring of right quotients of B) ([1]). B is canonically a subring of A.

Lemma 7. For any J, $K \in \boldsymbol{D}$, we have

$$_B\mathrm{Hom}(J_B,B_B) \otimes {}_B\mathrm{Hom}(K_B,B_B)_B \cong {}_B\mathrm{Hom}(KJ_B,B_B)_B.$$

Proof. Since K_B is finitely generated and projective, we have

$$\begin{aligned} &{}_B\mathrm{Hom}(J_B,B_B) \otimes {}_B\mathrm{Hom}(K_B,B_B)_B \cong {}_B\mathrm{Hom}(K_B,\mathrm{Hom}(J_B,B_B) \otimes {}_B B_B)_B \\ &\cong {}_B\mathrm{Hom}(K_B,\mathrm{Hom}(J_B,B_B)_B)_B \cong {}_B\mathrm{Hom}(K \otimes {}_B J_B,\ B_B)_B \\ &\cong {}_B\mathrm{Hom}(KJ_B,\ B_B)_B \end{aligned}$$

$$\begin{aligned} &h_1 \otimes h_2 \to (x_1 \to h_1 \otimes h_2(x_1)) \\ &\to (x_1 \to h_1 \cdot h_2(x_1)) \to (x_1 \otimes x_2 \to h_1[h_2(x_1) \cdot x_2]) \\ &\to (x_1 \cdot x_2 \to [h_1 \cdot h_2](x_1 \cdot x_2)) = h_1 \cdot h_2. \end{aligned}$$

Proposition 8. Under the above situations, we have the map $A \otimes {}_B A \to A$ $(a_1 \otimes a_2 \to a_1 a_2)$ is an isomorphism and $_BA$ is flat.

Proof. Since any element of $\boldsymbol{D}$ is finitely generated and projective, by Lemma 7,

$$\begin{aligned} A \otimes {}_B A &= \varinjlim_{\boldsymbol{D}} \mathrm{Hom}(J_B,B_B) \otimes {}_B \varinjlim_{\boldsymbol{D}} \mathrm{Hom}(K_B,B_B) \cong \varinjlim_{\boldsymbol{D}\times\boldsymbol{D}} \mathrm{Hom}(K \otimes {}_B J,B_B) \\ &\cong \varinjlim_{\boldsymbol{D}\times\boldsymbol{D}} \mathrm{Hom}(KJ_B,B_B) \cong \varinjlim_{\boldsymbol{D}} \mathrm{Hom}(I_B,B_B) = A. \end{aligned}$$

Let $0 \to X_B \to Y_B$ is exact. Since $0 \to \mathrm{Hom}(J_B,X_B) \to \mathrm{Hom}(J_B,Y_B)$ is exact and J_B is finitely generated and projective, $0 \to X \otimes {}_B\mathrm{Hom}(J_B,B_B) \to Y \otimes {}_B\mathrm{Hom}(J_B,B_B)$ is exact. Hence, $0 \to X \otimes {}_B \varinjlim_{\boldsymbol{D}} \mathrm{Hom}(J_B,B_B) \to Y \otimes {}_B \varinjlim_{\boldsymbol{D}} \mathrm{Hom}(J_B,B_B)$ is exact.

Lemma 9. Let $J \in \boldsymbol{D}$ and $J' = \{b' \in B' \mid b'V \subseteq VJ\}$ be the ideal of B' which corresponds to J. Then, J' is dense as a right ideal of B' and

$$\begin{aligned} &{}_{B'}V \otimes {}_B\mathrm{Hom}(J_B,B_B) \otimes {}_B V^*{}_{B'} \cong {}_{B'}\mathrm{Hom}(J'_{B'},B'_{B'})_{B'} \\ &\tau \otimes \delta \otimes \xi \to (y \to \Sigma[\tau \cdot \{\delta \cdot \xi(v_j)\}(x_j)]\psi'_j) \end{aligned}$$

where $J' \cong {}_{B'}V \otimes {}_B J \otimes {}_B V^*{}_{B'} = {}_{B'}V \otimes {}_B J \otimes {}_B\mathrm{Hom}({}_{B'}V,{}_{B'}B')_{B'}$ $(y \to \Sigma v_j \otimes x_j \otimes \psi'_j)$ $(y \in J',\ v_j \in V,\ x_j \in J,\ \psi'_j \in V^*)$. In this case, $y = \Sigma(v_j x_j)\psi'_j$.

Proof. Since V_B is finitely generated and projective,

$$\begin{aligned} J' &\cong {}_{B'}\mathrm{Hom}(V_B,VJ_B)_{B'} \cong {}_{B'}VJ \otimes {}_B\mathrm{Hom}(V_B,B_B)_{B'} \\ &\cong {}_{B'}V \otimes {}_B J \otimes {}_B\mathrm{Hom}(V_B,B_B)_{B'} \cong {}_{B'}V \otimes {}_B J \otimes {}_B\mathrm{Hom}({}_{B'}V,{}_{B'}B')_{B'}, \\ &\cong {}_{B'}\mathrm{Hom}({}_BV^*,{}_BJV^*)_{B'} \end{aligned}$$

$$\begin{aligned} &y \to (v \to yv) \to \Sigma v_j x_j \otimes \psi_j \\ &\to \Sigma v_j \otimes x_j \otimes \psi_j \to \Sigma v_j \otimes x_j \otimes \psi'_j \\ &\to (g \to \Sigma g(v_j)x_j\psi_j). \end{aligned}$$

In this case, by Lemma 3, for any $v \in V$, $yv = \Sigma v_j \cdot x_j \cdot \psi_j\,(v) = \Sigma(v_j \cdot x_j)\psi'_j \cdot v$.
Hence $y = \Sigma(v_j x_j)\psi'_j$. Moreover, since $[\Sigma g(v_j)x_j\psi_j](v) = \Sigma g(v_j)x_j\psi_j(v) = g(\Sigma v_j x_j \psi_j(v))$ $= g(\Sigma(v_j x_j)\psi'_j\ v) = g(yv) = (gy)(v)$, we have $\Sigma g(v_j)x_j\psi_j = gy$.

Let $b'J' = 0$. For any $v \in V$ and $w \in JV^*$, we can define the map $\eta_{v,w}\colon {}_BV^* \to {}_BJV^*$ $(g \to g(v)w)$. Then, for any $g \in V^*$, we have $0 = (g)[b'\eta_{v,w}] = (gb')\eta_{v,w} = (gb')(v)w = g(b'v)w$. Since $_BJV^*$ is faithful, $g(b'v) = 0$. Hence $b'v = 0$ and $b' = 0$. Further,

$$
\begin{aligned}
&{}_{B'}V \otimes_B \mathrm{Hom}(J_B,B_B) \otimes_B V^*{}_{B'} = {}_{B'}V \otimes_B \mathrm{Hom}(J_B,B_B) \otimes_B \mathrm{Hom}(V_B,B_B)_{B'} \\
&\cong {}_{B'}V \otimes_B \mathrm{Hom}(V_B,\mathrm{Hom}(J_B,B_B)_B)_{B'} \cong {}_{B'}V \otimes_B \mathrm{Hom}(V \otimes_B J_B,B_B)_{B'} \\
&\cong {}_{B'}\mathrm{Hom}(V \otimes_B J_B,V_B)_{B'} \cong {}_{B'}\mathrm{Hom}(V \otimes_B J_B,\mathrm{Hom}(V^*{}_{B'},B'{}_{B'})_B)_{B'} \\
&\cong {}_{B'}\mathrm{Hom}(V \otimes_B J \otimes_B V^*{}_{B'},B'{}_{B'})_{B'} \cong {}_{B'}\mathrm{Hom}(J'{}_{B'},B'{}_{B'})_{B'}.
\end{aligned}
$$

Theorem 10. Let $A = \varinjlim_{\boldsymbol{D}} \mathrm{Hom}(J_B,B_B)$ (a ring of right quotients of B), $U = V \otimes_B A$, and $A' = \mathrm{End}(U_A)$.

And let $\boldsymbol{D}' = \{J' \subseteq B' \mid J'$ is an ideal of B' and dense as a right ideal$\}$. Then

$A' \cong \varinjlim_{\boldsymbol{D}'} \mathrm{Hom}(J'{}_{B'},B'{}_{B'})$ (a ring of right quotient of B') and ${}_{B'}A'$ is flat.

Proof. By Lemma 9, we have

$$
\begin{aligned}
&{}_{B'}\varinjlim_{\boldsymbol{D}'}\mathrm{Hom}(J'{}_{B'},B'{}_{B'})_{B'} \cong {}_{B'}V \otimes_B \varinjlim_{\boldsymbol{D}}\mathrm{Hom}(J_B,B_B) \otimes_B V^*{}_{B'} \\
&\cong {}_{B'}V \otimes_B A \otimes_B V^*{}_{B'} = {}_{B'}U \otimes_B \mathrm{Hom}(V_B,B_B)_{B'} \\
&\cong {}_{B'}\mathrm{Hom}(V_B,U_B)_{B'} \\
&\cong {}_{B'}\mathrm{Hom}(V_B,A' \otimes_{B'} V_B)_{B'} \cong {}_{B'}A' \otimes_{B'} \mathrm{Hom}(V_B,V_B)_{B'} \cong {}_{B'}A'{}_{B'}.
\end{aligned}
$$

Then, in the above isomorphisms, we will show that 1 of $\varinjlim_{\boldsymbol{D}'} \mathrm{Hom}(J'{}_{B'},B'{}_{B'})$ corresponds to 1 of A'. Since V_B is finitely generated and projective, there exist

$\tau_k \in V$ and $\xi_k \in \mathrm{Hom}(V_B,B_B)$ such that for any $v \in V$, $v = \sum \tau_k \cdot \xi_k(v)$. Then,

$$
\begin{aligned}
&[1] = [(y \to \sum_{j,k}[\tau_k \cdot \xi_k(v_j) \cdot x_j]\psi'_j = y)] \leftarrow \sum \tau_k \otimes [1] \otimes \xi_k \\
&\to \sum \tau_k \otimes 1 \otimes \xi_k = \sum(\tau_k \otimes 1) \otimes \xi_k \\
&\to (v \to \sum(\tau_k \otimes 1)\cdot \xi_k(v) = \sum \tau_k \cdot \xi_k(v) \otimes 1 = v \otimes 1_A = \phi'(a' \otimes v)) \\
&\leftarrow (v \to a' \otimes v) \leftarrow a' \otimes 1 \leftarrow a'.
\end{aligned}
$$

In this case, by Lemma 1, for any $a \in A$, $v \otimes a = (v \otimes 1_A)a = \phi'(a' \otimes v)a = a'\phi'(1_{A'} \otimes v)a = a'(v \otimes 1_A)a$ $= a'(v \otimes a)$. Hence $a' = 1$. That is, the isomorhism $\varinjlim_{\boldsymbol{D}'}\mathrm{Hom}(J'{}_{B'},B'{}_{B'})_{B'} \to A'$ is an isomorphism as rings.

The fact that ${}_{B'}A'$ is flat follows from Proposition 8 and Proposition 5.

Proposition 11. In Theorem 10, we have

$\{M'{}_{B'} \mid M'{}_{B'} \cong M \otimes_B V^*{}_{B'}$ for some M_B such that $M \otimes_B A = 0\} = \{M'{}_{B'} \mid M' \otimes_{B'} A' = 0\}$.

Proof. Let $M \otimes_B A = 0$. Since ${}_{B'}V$ and U_A are finitely genarared and projective, by Lemma 4 (2), $M \otimes_B V^*{}_{B'} \otimes_{B'} A' \cong M \otimes_B \mathrm{Hom}({}_{B'}V,{}_{B'}B') \otimes_{B'} A' \cong M \otimes_B \mathrm{Hom}({}_{B'}V,{}_{B'}A') \cong M \otimes_B U^* \cong M \otimes_B \mathrm{Hom}(U_A,A_A)$ $\cong \mathrm{Hom}(U_A,M \otimes_B A_A) = 0$.

Convesely, let $M' \otimes_{B'} A' = 0$ and put $M_B = M' \otimes_{B'} V_B$. Then, we have $M \otimes_B A = M' \otimes_{B'} V \otimes_B A$ $\cong M' \otimes_{B'} U \cong M' \otimes_{B'} A' \otimes_{A'} U = 0$ and $M' \otimes_{B'} V \otimes_B V^*{}_{B'} \cong M'{}_{B'}$.

(*) In this case, $\boldsymbol{T} = \{M_B \mid M \otimes_B A = 0\}$ is a hereditary torsin class,

$\boldsymbol{F} = \{\mathfrak{A}_B \mid \mathfrak{A}_B$ is a right ideal of B such that $\mathfrak{A}A = A\}$

$= \{\mathfrak{A}_B \mid \mathfrak{A}_B$ is a right ideal of B such that $J \subseteq \mathfrak{A}$ for some $J \in \boldsymbol{D}\}$

is a topology and $A \cong \varinjlim_{\boldsymbol{F}}\mathrm{Hom}(\mathfrak{A}_B, B_B)$ ([8], p.78).

Moreover, $\boldsymbol{T}' = \{M'{}_{B'} \mid M'{}_{B'} \cong M \otimes_B V^*{}_{B'}$ for some $M_B \in \boldsymbol{T}\}$

$= \{M'{}_{B'} \mid M' \otimes_{B'} A' = 0\}$

is a hereditary torsion class,

$\boldsymbol{F}' = \{\mathfrak{A}'{}_{B'} \mid \mathfrak{A}'{}_{B'}$ is a right ideal of B' such that $\mathfrak{A}'A' = A'\}$

$= \{\mathfrak{A}'{}_{B'} \mid \mathfrak{A}'{}_{B'}$ is a right ideal of B' such that $J' \subseteq \mathfrak{A}'$ for some $J' \in \boldsymbol{D}'\}$

and $A' \cong \varinjlim_{F} \mathrm{Hom}(\mathfrak{A}'_{B'}, B'_{B'})$ ([4], p.663).

要　　約

森田加群に関連して，ある種の商環の森田同値性について調べた。

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